Wolfe's Combinatorial Method is Exponential

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Computational and Applied Mathematics, UCLA


joint with Jesús De Loera and Luis Rademacher (UC Davis)
https://arxiv.org/abs/1710.02608

Minimum Norm Point (MNP(P))

Minimum Norm Point in Polytope

We are interested in solving the problem (MNP(P)):

$$\min_{\mathbf{x} \in P} \|\mathbf{x}\|_2$$

where P is a polytope, and determining the minimum dimension face, F, which achieves distance $\|\mathbf{x}\|_2$.

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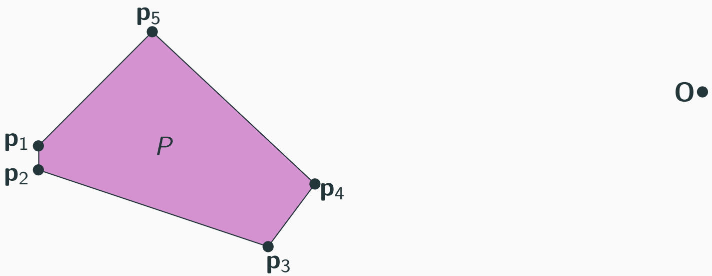
where P is a polytope, and determining the minimum dimension face, F, which achieves distance $\|\mathbf{x}\|_2$.

Note: We consider polytopes, P, given in V-representation as the convex hull of points $\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m$,

$$P = \left\{ \sum_{i=1}^{m} \lambda_i \mathbf{p}_i : \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0 \text{ for all } i = 1, 2, ..., m \right\}.$$

Minimum Norm Point in Polytope



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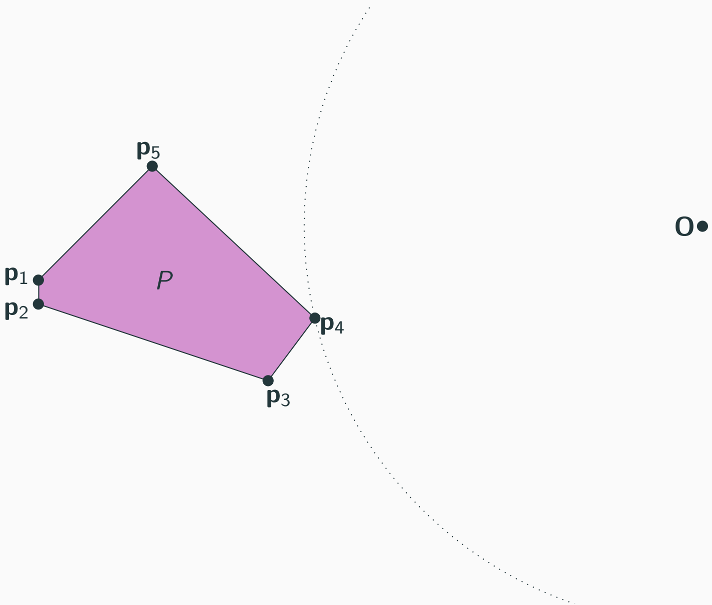

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`permits combinatorial algorithms`

Applications

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- compute distance to polytope

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If a strongly-polynomial method for projection onto a polytope exists then this gives a strongly-polynomial method for LP.

It was previously known that linear programming reduces to MNP on a polytope in weakly-polynomial time [Fujishige, Hayashi, Isotani '06].

Spoiler

Theorem (De Loera, H., Rademacher ’17)

There exists a family of polytopes on which Wolfe’s method requires exponential time to compute the MNP.

Wolfe's Optimality Condition

Theorem (Wolfe '74)

Let $P = conv(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m)$*. Then* $\mathbf{x} \in P$ *is* $MNP(P)$ *if and only if*

$$\mathbf{x}^T \mathbf{p}_j \geq \|\mathbf{x}\|_2^2 \text{ for all } j = 1, 2, ..., m.$$

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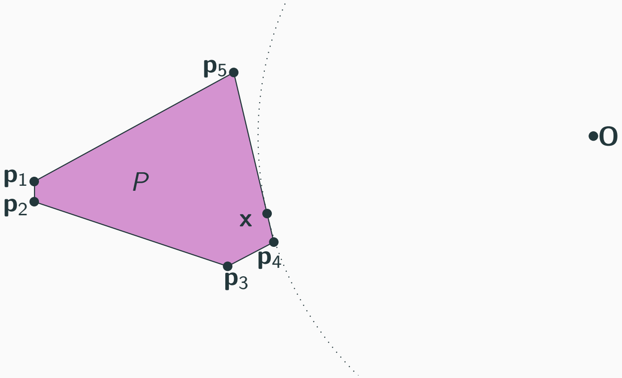

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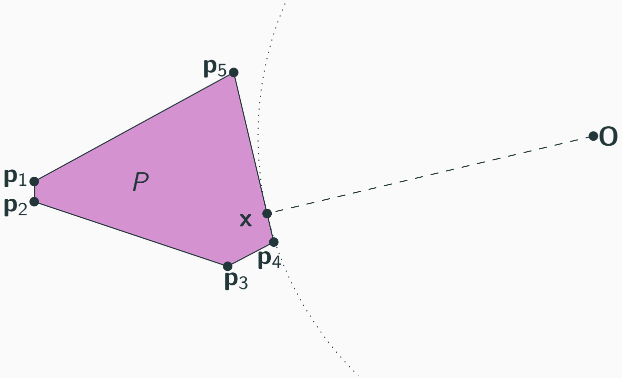

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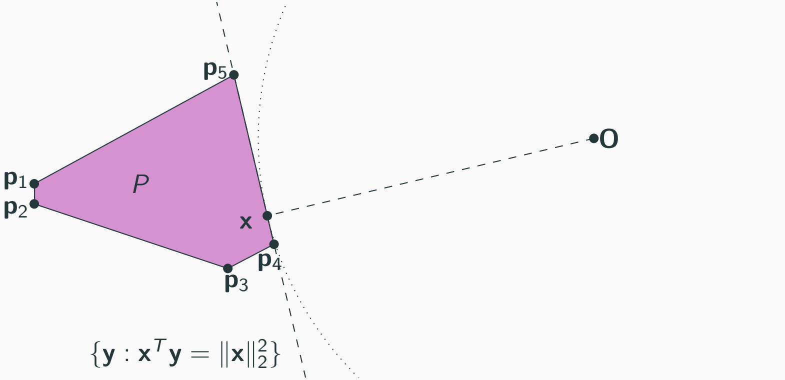
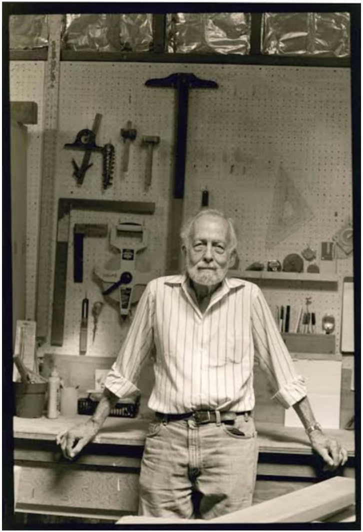


Wolfe's Method

Philip Wolfe



- Frank-Wolfe method
- Dantzig-Wolfe decomposition
- simplex method for quadratic programming

Intuition and Definitions

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Def: An affinely independent set of points $Q = \{\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_k\}$ is a *corral* if $\text{MNP}(\text{aff}(Q)) \in \text{relint}(\text{conv}(Q))$.

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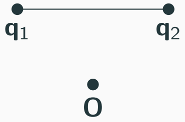

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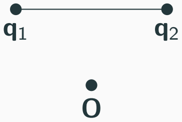

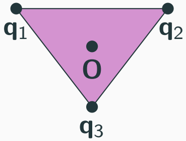

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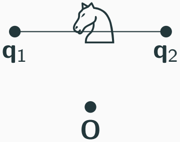

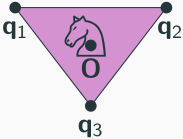

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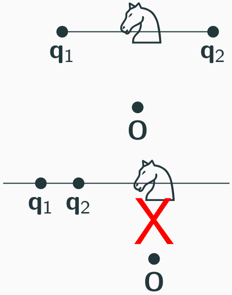

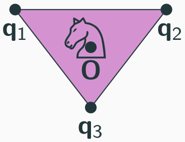

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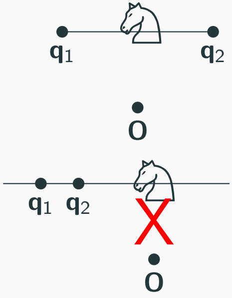

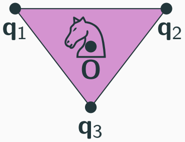

Note: Singletons are corrals.

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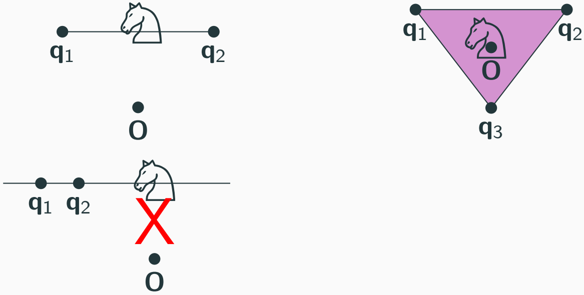

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Note: There is a corral in P whose convex hull contains $\text{MNP}(P)$.

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Wolfe's method : combinatorial method for computing projection onto a vertex-representation polytope

- pivots between corrals which may contain MNP(P)
- projects onto affine hull of sets to check whether a corral
- optimality criterion checks if correct corral

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
    $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
    $C = C \cup \{\mathbf{p}_j\}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
        $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
        $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
        $\mathbf{x} = \mathbf{z}$
        $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
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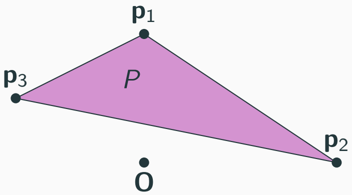

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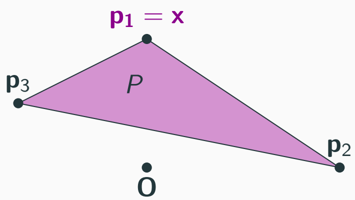

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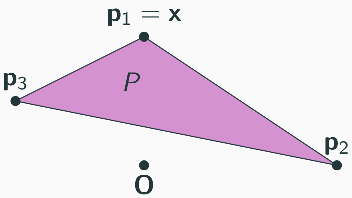

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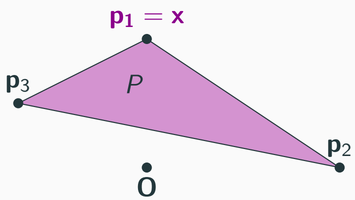

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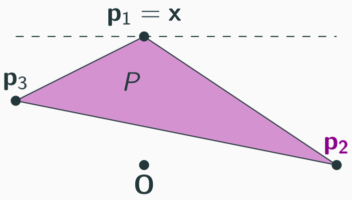

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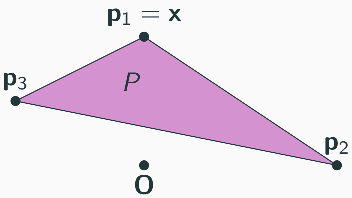

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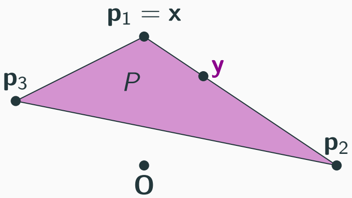

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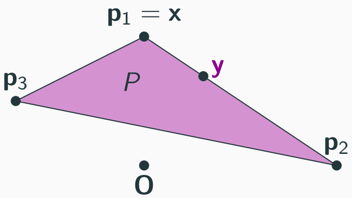

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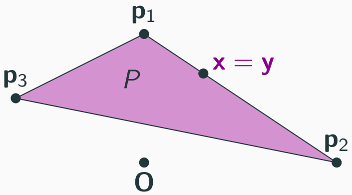
    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
    $\mathbf{x} = \mathbf{z}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$

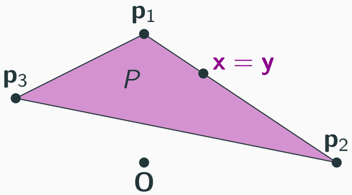


$C = \{\mathbf{p}_1, \mathbf{p}_2\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while x is not MNP(P)
  $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
  $C = C \cup \{\mathbf{p}_j\}$
  $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
    $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of $\text{conv}(C)$
    $\mathbf{x} = \mathbf{z}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_1, \mathbf{p}_2\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$

$C = \{\mathbf{x}\}$

while $\mathbf{x}$ is not MNP(P)

  $\mathbf{p_j} \in \{\mathbf{p} \in \mathbf{P} : \mathbf{x}^\mathbf{T}\mathbf{p} < ||\mathbf{x}||_2^2\}$

  $C = C \cup \{\mathbf{p}_j\}$

  $\mathbf{y} = \text{MNP}(\text{aff}(C))$

  while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$

    $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$

    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)

    $\mathbf{x} = \mathbf{z}$

    $\mathbf{y} = \text{MNP}(\text{aff}(C))$

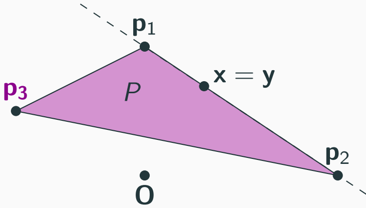
  $\mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$

$\mathbf{p}_2 = (3, 0)$

$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_1, \mathbf{p}_2\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$

$C = \{\mathbf{x}\}$

while $\mathbf{x}$ is not MNP(P)

$\quad \mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$

$\quad \mathbf{C} = \mathbf{C} \cup \{\mathbf{p_j}\}$

$\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

$\quad$ while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$

$\quad\quad \mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$

$\quad\quad C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)

$\quad\quad \mathbf{x} = \mathbf{z}$

$\quad\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

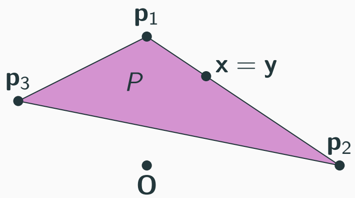
$\quad \mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$

$\mathbf{p}_2 = (3, 0)$

$\mathbf{p}_3 = (-2, 1)$

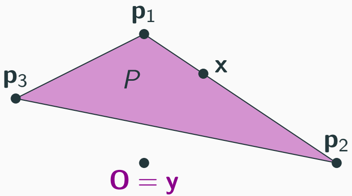


$\mathbf{C} = \{\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
    $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
    $C = C \cup \{\mathbf{p}_j\}$
    $\mathbf{y} = \mathbf{MNP(aff(C))}$
    while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
        $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
        $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
        $\mathbf{x} = \mathbf{z}$
        $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$

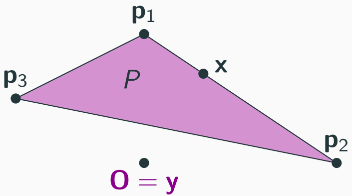


$C = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
    $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
    $C = C \cup \{\mathbf{p}_j\}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    while y $\notin$ relint(conv(C))
        $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
        $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
        $\mathbf{x} = \mathbf{z}$
        $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$

$C = \{\mathbf{x}\}$

while $\mathbf{x}$ is not MNP(P)

$\quad \mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$

$\quad C = C \cup \{\mathbf{p}_j\}$

$\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

$\quad$ while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$

$\qquad \mathbf{z} = \underset{\mathbf{z} \in \mathbf{conv}(\mathbf{C}) \cap \overline{\mathbf{xy}}}{\textbf{argmin}} ||\mathbf{z} - \mathbf{y}||_2$

$\qquad C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)

$\qquad \mathbf{x} = \mathbf{z}$

$\qquad \mathbf{y} = \text{MNP}(\text{aff}(C))$

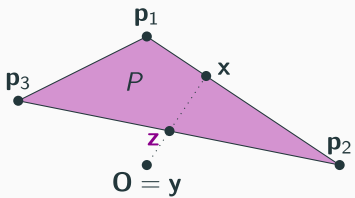
$\quad \mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$

$\mathbf{p}_2 = (3, 0)$

$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$

$C = \{\mathbf{x}\}$

while $\mathbf{x}$ is not MNP(P)

$\quad \mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$

$\quad C = C \cup \{\mathbf{p}_j\}$

$\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

$\quad$ while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$

$\quad\quad \mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$

$\quad\quad$ **$\mathbf{C} = \mathbf{C} - \{\mathbf{p_i}\}$ where $\mathbf{p_i}, \mathbf{z}$ are on different faces of conv($\mathbf{C}$)**

$\quad\quad \mathbf{x} = \mathbf{z}$

$\quad\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

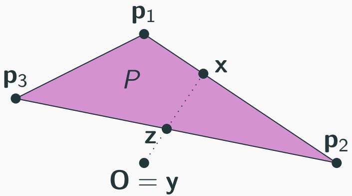
$\quad \mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$

$\mathbf{p}_2 = (3, 0)$

$\mathbf{p}_3 = (-2, 1)$

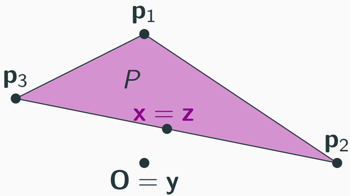


$\mathbf{C} = \{\mathbf{p_2}, \mathbf{p_3}\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
  $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
  $C = C \cup \{\mathbf{p}_j\}$
  $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
    $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of $\text{conv}(C)$
    $\mathbf{x} = \mathbf{z}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$

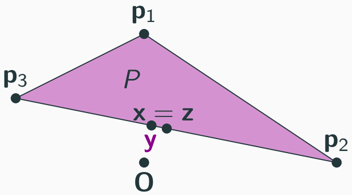


$C = \{\mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
  $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
  $C = C \cup \{\mathbf{p}_j\}$
  $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
    $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
    $\mathbf{x} = \mathbf{z}$
    $\mathbf{y} = \textbf{MNP}(\textbf{aff}(\mathbf{C}))$
  $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$

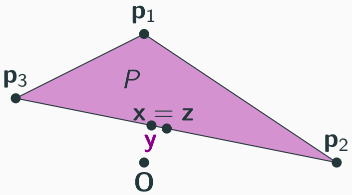


$C = \{\mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
  $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
  $C = C \cup \{\mathbf{p}_j\}$
  $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  while $\mathbf{y} \notin$ relint(conv(C))
    $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
    $\mathbf{x} = \mathbf{z}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$

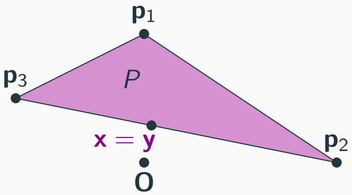


$C = \{\mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
  $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
  $C = C \cup \{\mathbf{p}_j\}$
  $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
    $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
    $\mathbf{x} = \mathbf{z}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$

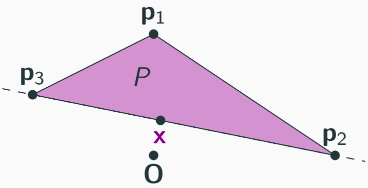


$C = \{\mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while x is not MNP(P)
    $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
    $C = C \cup \{\mathbf{p}_j\}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
        $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
        $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of $\text{conv}(C)$
        $\mathbf{x} = \mathbf{z}$
        $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$

$C = \{\mathbf{x}\}$

while $\mathbf{x}$ is not MNP(P)

$\quad \mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$

$\quad C = C \cup \{\mathbf{p}_j\}$

$\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

$\quad$ while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$

$\quad\quad \mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$

$\quad\quad C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)

$\quad\quad \mathbf{x} = \mathbf{z}$

$\quad\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

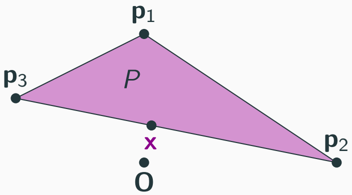
$\quad \mathbf{x} = \mathbf{y}$

return x

$\mathbf{p}_1 = (0, 2)$

$\mathbf{p}_2 = (3, 0)$

$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_2, \mathbf{p}_3\}$

Wolfe's Method

$\mathbf{x} = \mathbf{p}_i$ for some $i = 1, 2, ..., m$, $\lambda = \mathbf{e}_i$

$C = \{i\}$

while $\mathbf{x} \neq \mathbf{0}$ and there exists $\mathbf{p}_j$ with $\mathbf{x}^T \mathbf{p}_j < \|\mathbf{x}\|_2^2$

$\quad C = C \cup \{j\}$

$\quad \alpha = \underset{\sum_{i \in C} \alpha_i = 1}{\operatorname{argmin}} \|\sum_{i \in C} \alpha_i \mathbf{p}_i\|_2$, $\mathbf{y} = \sum_{i \in C} \alpha_i \mathbf{p}_i$

$\quad$ while $\alpha_i \leq 0$ for some $i = 1, 2, ..., m$

$\quad\quad \theta = \min_{i : \alpha_i \leq 0} \frac{\lambda_i}{\lambda_i - \alpha_i}$

$\quad\quad \mathbf{z} = \theta \mathbf{y} + (1 - \theta) \mathbf{x}$

$\quad\quad i \in \{j : \theta \alpha_j + (1 - \theta) \lambda_j = 0\}$

$\quad\quad C = C - \{i\}$

$\quad\quad \mathbf{x} = \mathbf{z}$

$\quad\quad$ solve $\mathbf{x} = P\lambda$ for λ

$\quad\quad \alpha = \underset{\sum_{i \in C} \alpha_i = 1}{\operatorname{argmin}} \|\sum_{i \in C} \alpha_i \mathbf{p}_i\|_2$, $\mathbf{y} = \sum_{i \in C} \alpha_i \mathbf{p}_i$

$\quad \mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

Wolfe's Method

$\mathbf{x} = \mathbf{p}_i$ for some $i = 1, 2, ..., m$, $\lambda = \mathbf{e}_i$

$C = \{i\}$

while $\mathbf{x} \neq \mathbf{0}$ and there exists $\mathbf{p}_j$ with $\mathbf{x}^T \mathbf{p}_j < \|\mathbf{x}\|_2^2$

$\quad C = C \cup \{j\}$

$\quad \alpha = \underset{\sum_{i \in C} \alpha_i = 1}{\text{argmin}} \|\sum_{i \in C} \alpha_i \mathbf{p}_i\|_2$, $\mathbf{y} = \sum_{i \in C} \alpha_i \mathbf{p}_i$

$\quad$ while $\alpha_i \leq 0$ for some $i = 1, 2, ..., m$

$\quad\quad \theta = \min_{i : \alpha_i \leq 0} \frac{\lambda_i}{\lambda_i - \alpha_i}$

$\quad\quad \mathbf{z} = \theta \mathbf{y} + (1 - \theta)\mathbf{x}$

$\quad\quad i \in \{j : \theta \alpha_j + (1 - \theta)\lambda_j = 0\}$

$\quad\quad C = C - \{i\}$

$\quad\quad \mathbf{x} = \mathbf{z}$

$\quad\quad$ solve $\mathbf{x} = P\lambda$ for λ

$\quad\quad \alpha = \underset{\sum_{i \in C} \alpha_i = 1}{\text{argmin}} \|\sum_{i \in C} \alpha_i \mathbf{p}_i\|_2$, $\mathbf{y} = \sum_{i \in C} \alpha_i \mathbf{p}_i$

$\quad \mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

Choice 1: Initial vertex.

Wolfe's Method

$\mathbf{x} = \mathbf{p}_i$ for some $i = 1, 2, ..., m$, $\lambda = \mathbf{e}_i$
$C = \{i\}$
while $\mathbf{x} \neq \mathbf{0}$ and there exists $\mathbf{p}_j$ with $\mathbf{x}^T\mathbf{p}_j < \|\mathbf{x}\|_2^2$
    $C = C \cup \{j\}$
    $\alpha = \underset{\sum_{i \in C} \alpha_i = 1}{\operatorname{argmin}} \|\sum_{i \in C} \alpha_i \mathbf{p}_i\|_2$, $\mathbf{y} = \sum_{i \in C} \alpha_i \mathbf{p}_i$
    while $\alpha_i \leq 0$ for some $i = 1, 2, ..., m$
        $\theta = \min_{i : \alpha_i \leq 0} \frac{\lambda_i}{\lambda_i - \alpha_i}$
        $\mathbf{z} = \theta\mathbf{y} + (1 - \theta)\mathbf{x}$
        $i \in \{j : \theta\alpha_j + (1 - \theta)\lambda_j = 0\}$
        $C = C - \{i\}$
        $\mathbf{x} = \mathbf{z}$
        solve $\mathbf{x} = P\lambda$ for λ
        $\alpha = \underset{\sum_{i \in C} \alpha_i = 1}{\operatorname{argmin}} \|\sum_{i \in C} \alpha_i \mathbf{p}_i\|_2$, $\mathbf{y} = \sum_{i \in C} \alpha_i \mathbf{p}_i$
    $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

Choice 1: Initial vertex.

Choice 2: Adding to corral.

Wolfe's Method

$\mathbf{x} = \mathbf{p}_i$ for some $i = 1, 2, ..., m$, $\lambda = \mathbf{e}_i$

$C = \{i\}$

while $\mathbf{x} \neq \mathbf{0}$ and there exists $\mathbf{p}_j$ with $\mathbf{x}^T \mathbf{p}_j < \|\mathbf{x}\|_2^2$

$\quad C = C \cup \{j\}$

$\quad \alpha = \underset{\sum_{i \in C} \alpha_i = 1}{\operatorname{argmin}} \|\sum_{i \in C} \alpha_i \mathbf{p}_i\|_2$, $\mathbf{y} = \sum_{i \in C} \alpha_i \mathbf{p}_i$

$\quad$ while $\alpha_i \leq 0$ for some $i = 1, 2, ..., m$

$\quad\quad \theta = \min_{i : \alpha_i \leq 0} \frac{\lambda_i}{\lambda_i - \alpha_i}$

$\quad\quad \mathbf{z} = \theta \mathbf{y} + (1 - \theta) \mathbf{x}$

$\quad\quad i \in \{j : \theta \alpha_j + (1 - \theta) \lambda_j = 0\}$

$\quad\quad C = C - \{i\}$

$\quad\quad \mathbf{x} = \mathbf{z}$

$\quad\quad$ solve $\mathbf{x} = P\lambda$ for λ

$\quad\quad \alpha = \underset{\sum_{i \in C} \alpha_i = 1}{\operatorname{argmin}} \|\sum_{i \in C} \alpha_i \mathbf{p}_i\|_2$, $\mathbf{y} = \sum_{i \in C} \alpha_i \mathbf{p}_i$

$\quad \mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

Choice 1: Initial vertex.

Choice 2: Adding to corral.

Choice 3: Removing from corral.

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Initial: `minnorm`

Insertion: `linopt` (select $\mathbf{p}_j$ minimizing $\mathbf{x}^T\mathbf{p}_j$), `minnorm`

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- examples in which each insertion rule is better

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- von Neumann's algorithm for linear programming

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  ▷ pseudo-polynomial complexity

Exponential Behavior

Exponential Example

Goal : build family of instances on which the number of iterations of Wolfe's method is at least exponential in the dimension and number of points

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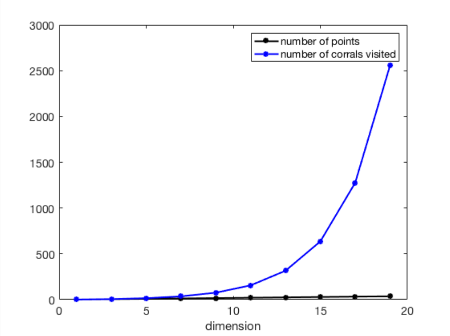

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$$\xrightarrow[+4 \text{ points}]{+2 \text{ dim}}$$

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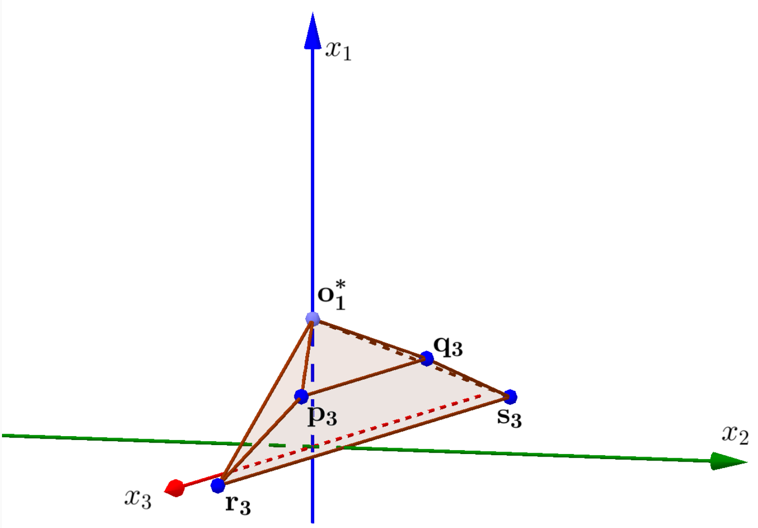
Exponential Example: dim 3



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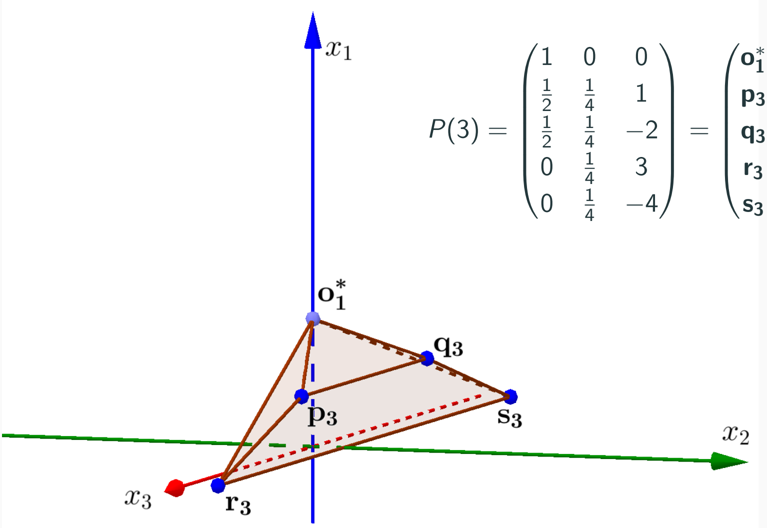

$$P(3) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 1 \\ \frac{1}{2} & \frac{1}{4} & -2 \\ 0 & \frac{1}{4} & 3 \\ 0 & \frac{1}{4} & -4 \end{pmatrix} = \begin{pmatrix} \mathbf{o}_1^* \\ \mathbf{p}_3 \\ \mathbf{q}_3 \\ \mathbf{r}_3 \\ \mathbf{s}_3 \end{pmatrix}$$

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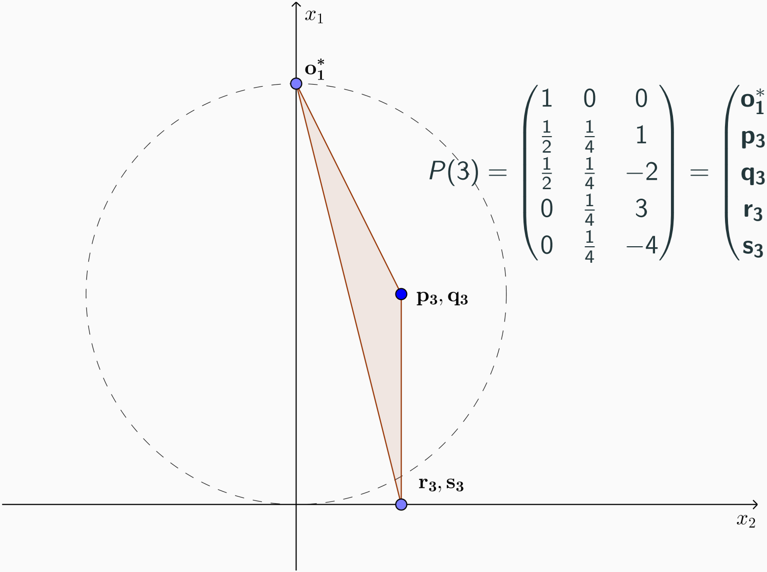

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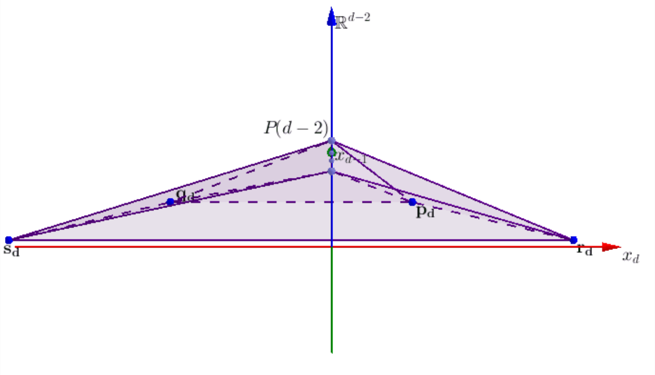

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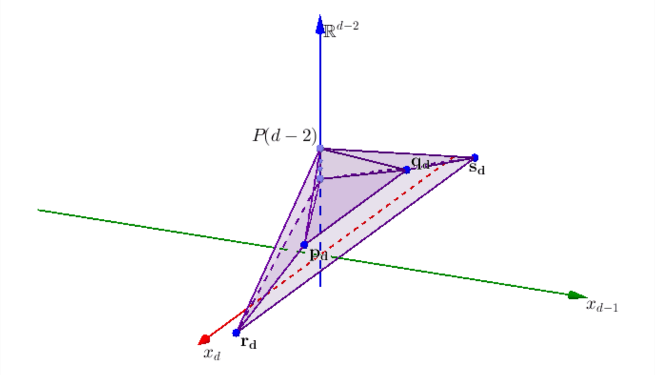

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Consider the execution of Wolfe's method with the $\mathtt{minnorm}$ *insertion rule on input* $P(d)$ *where* $d = 2k - 1$*. Then the sequence of corrals,* $C(d)$ *has length* $5 \cdot 2^{k-1} - 4$*.*

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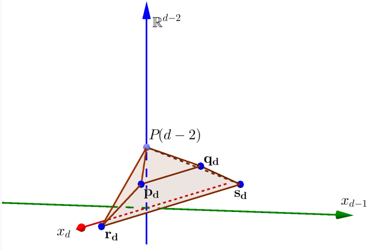

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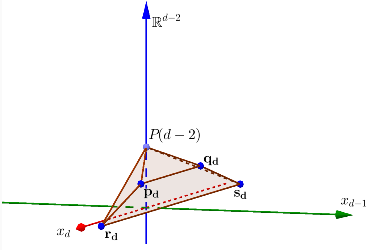

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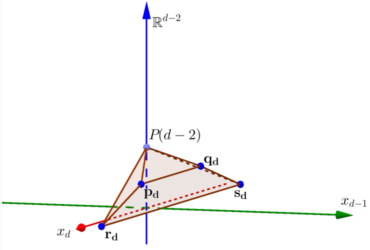

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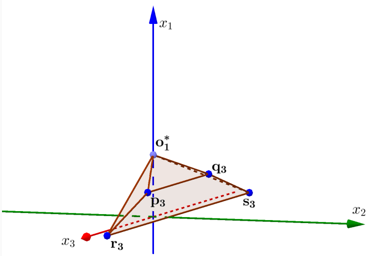

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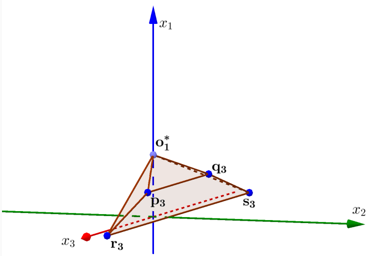

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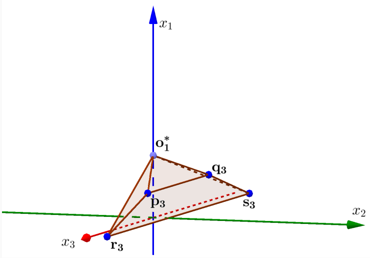

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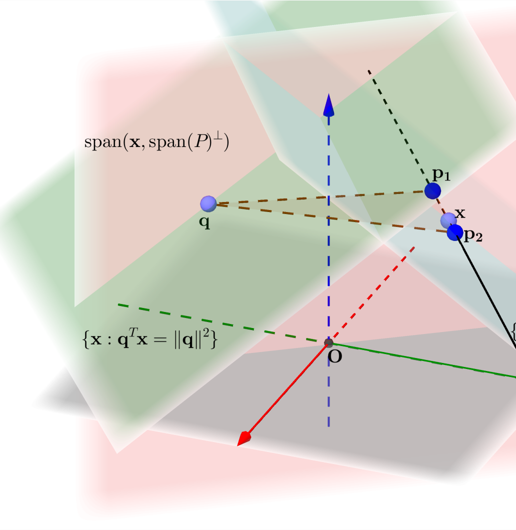

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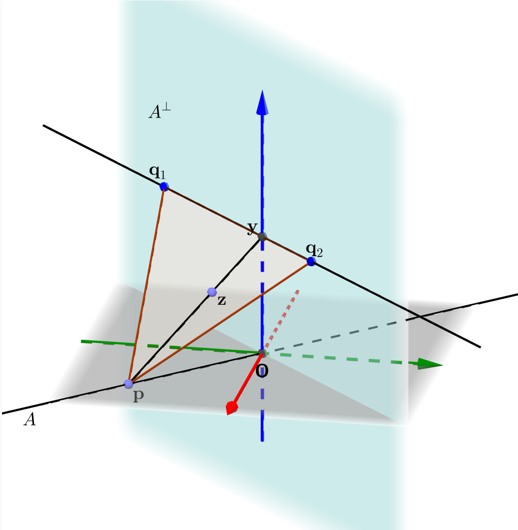

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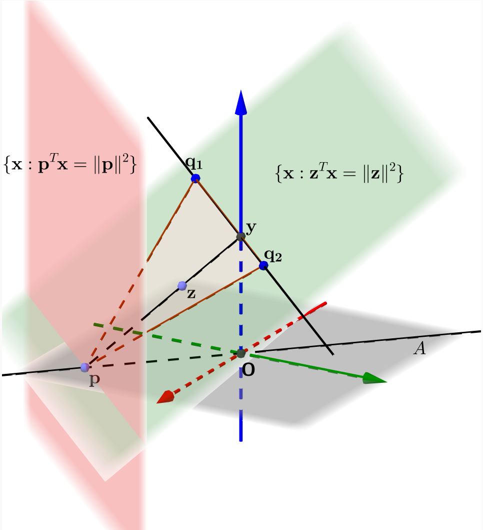
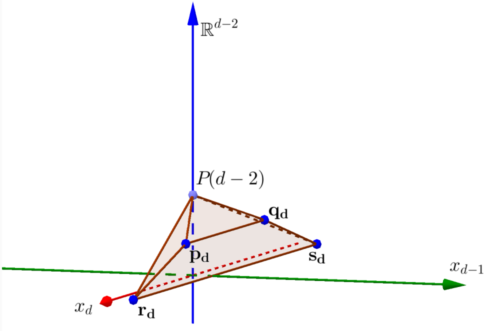


Sketch of Proof of Sequence $C(d)$: $C(d-2)$



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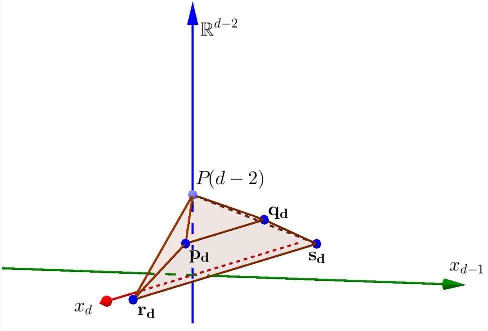

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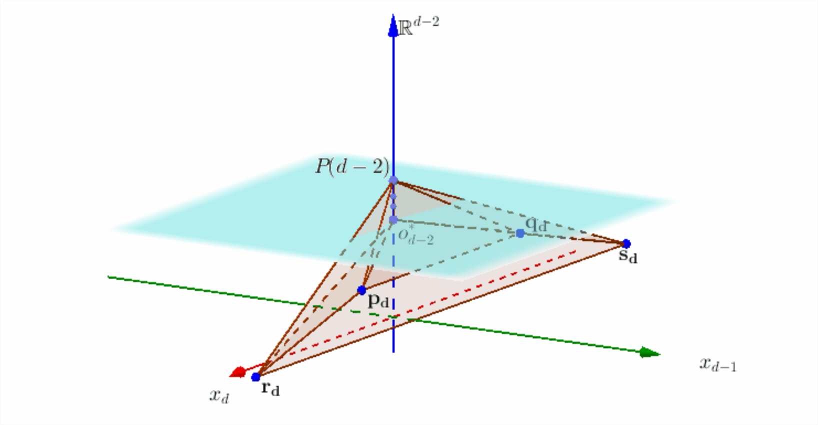

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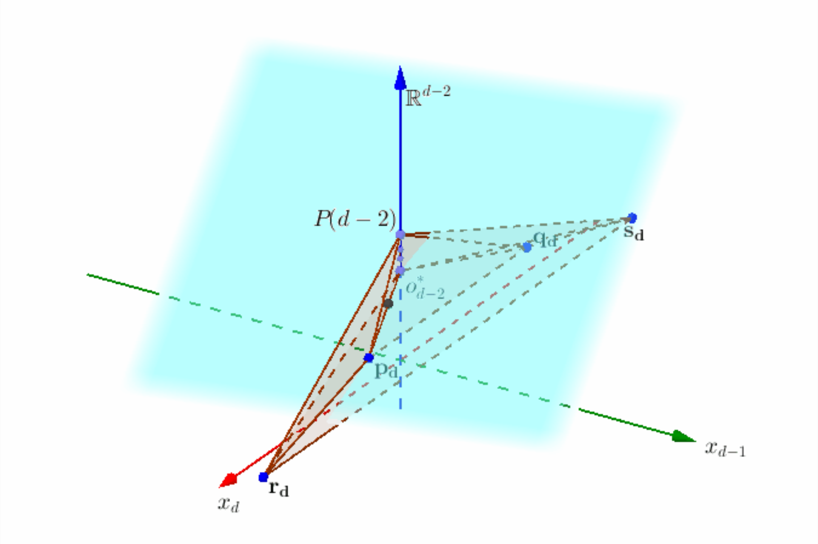

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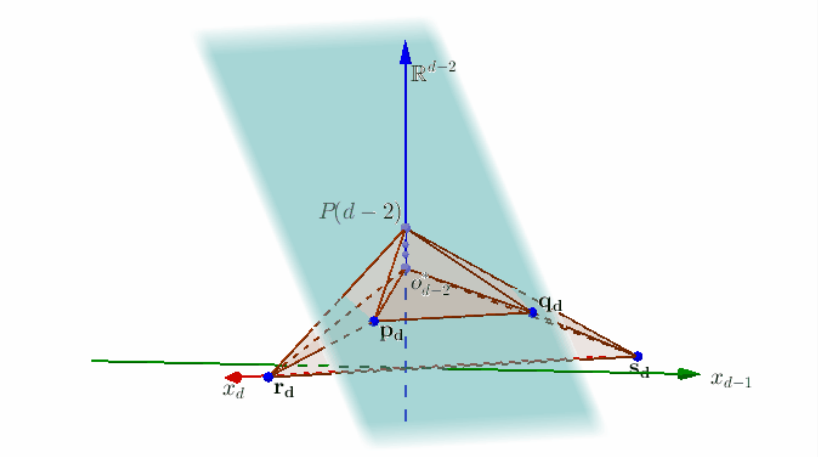

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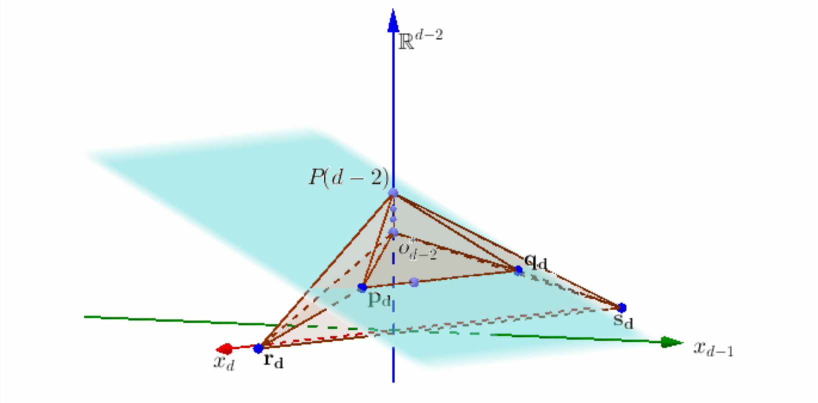

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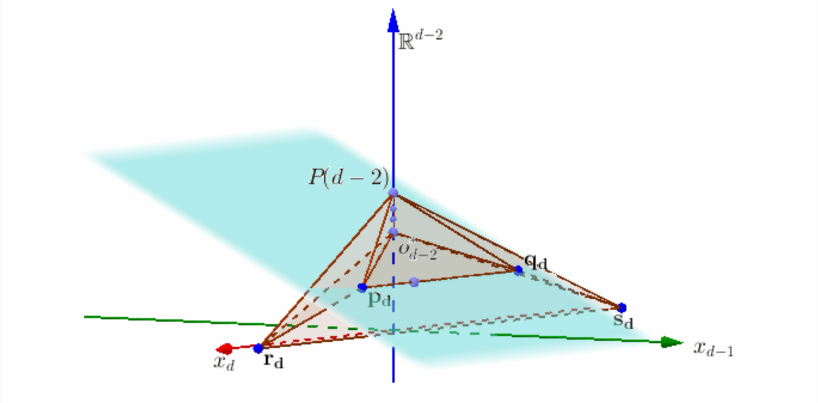

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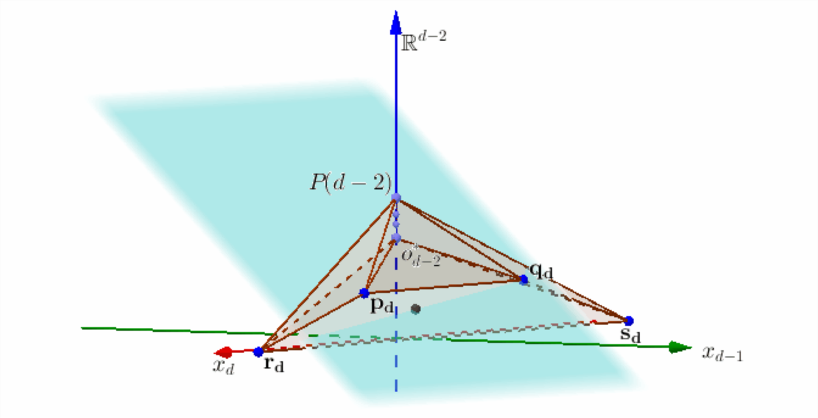

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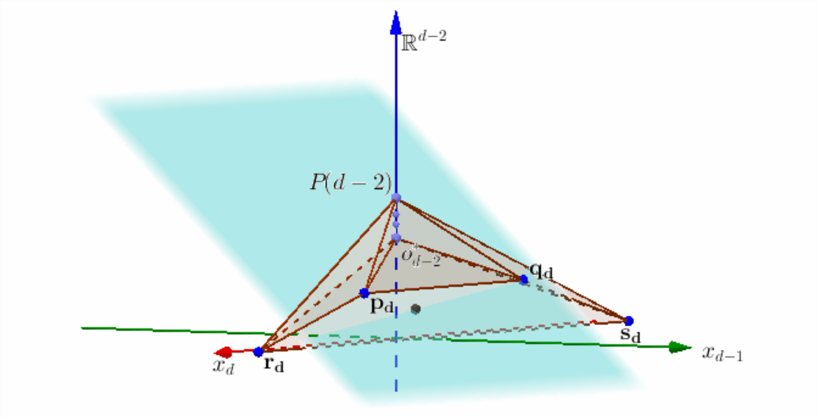

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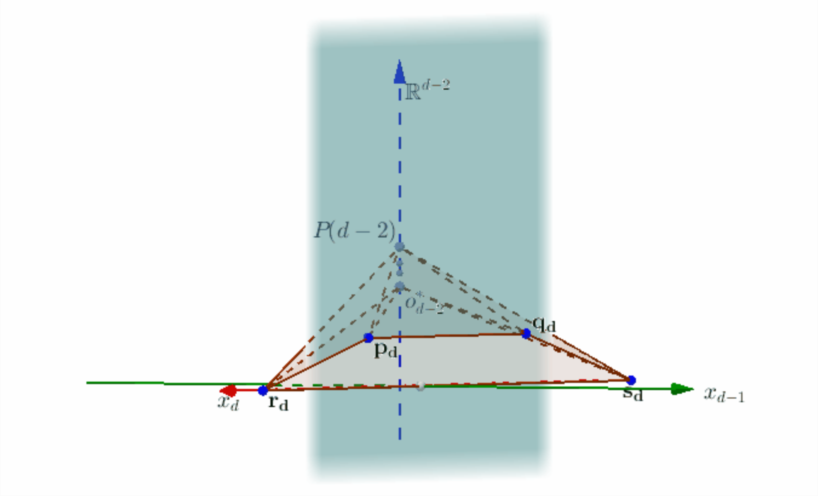

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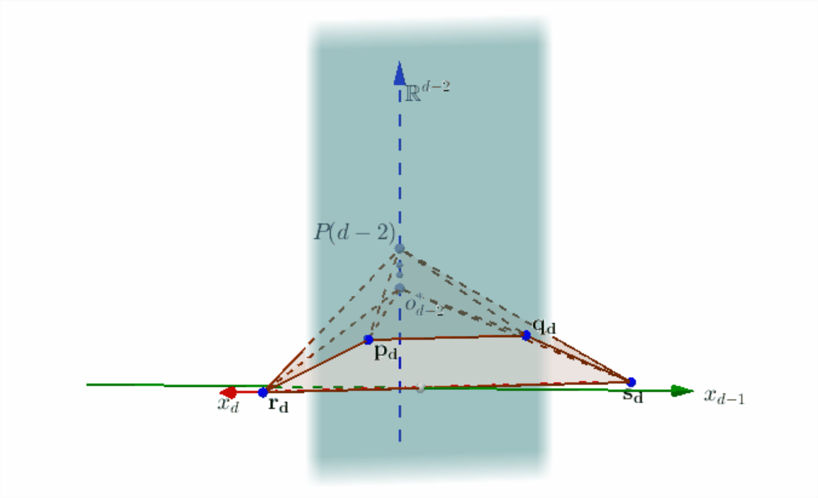

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4. Give an average (or smoothed) analysis of Wolfe's method.

Questions?

[1] I. Bárány and S. Onn.
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Mathematics of Operations Research, 22(3):550–567, 1997.

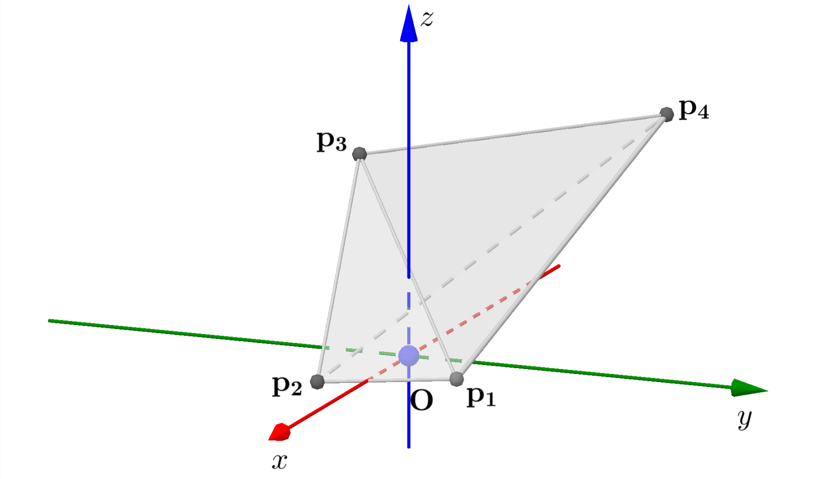
[2] D. Chakrabarty, P. Jain, and P. Kothari.
Provable submodular minimization using wolfe's algorithm.
CoRR, abs/1411.0095, 2014.

[3] J. A. De Loera, J. Haddock, and L. Rademacher.
The minimum Euclidean-norm point on a convex polytope: Wolfes combinatorial algorithm is exponential.
2017.

[4] S. Fujishige, T. Hayashi, and S. Isotani.
The minimum-norm-point algorithm applied to submodular function minimization and linear programming.
Citeseer, 2006.

Example: `minnorm` < `linopt`

$P = \mathrm{conv}\{(0.8, 0.9, 0), (1.5, -0.5, 0), (-1, -1, 2), (-4, 1.5, 2)\} \subset \mathbb{R}^3$



Example: `minnorm` $<$ `linopt`

Major Cycle	Minor Cycle	C
0	0	$\{p_1\}$
1	0	$\{p_1, p_2\}$
2	0	$\{p_1, p_2, p_3\}$
3	0	$\{p_1, p_2, p_3, p_4\}$
3	1	$\{p_1, p_2, p_4\}$

Major Cycle	Minor Cycle	C
0	0	$\{p_1\}$
1	0	$\{p_1, p_4\}$
2	0	$\{p_1, p_4, p_3\}$
2	1	$\{p_1, p_3\}$
3	0	$\{p_1, p_3, p_2\}$
4	0	$\{p_1, p_2, p_3, p_4\}$
4	1	$\{p_1, p_2, p_4\}$

Example: minnorm < linopt

Major Cycle	Minor Cycle	C
0	0	$\{p_1\}$
1	0	$\{p_1, p_2\}$
2	0	$\{p_1, p_2, p_3\}$
3	0	$\{p_1, p_2, p_3, p_4\}$
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minnorm < linopt

Major Cycle	Minor Cycle	C
0	0	$\{p_1\}$
1	0	$\{p_1, p_4\}$
2	0	$\{p_1, p_4, p_3\}$
2	1	$\{p_1, p_3\}$
3	0	$\{p_1, p_3, p_2\}$
4	0	$\{p_1, p_2, p_3, p_4\}$
4	1	$\{p_1, p_2, p_4\}$