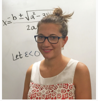
Greedy and Randomized Projection Methods

Jamie Haddock

UC Irvine Probability Seminar,

October 1, 2019

Computational and Applied Mathematics
UCLA



joint with Jesús A. De Loera, Deanna Needell, and Anna Ma

https://arxiv.org/abs/1802.03126 (BIT Numerical Mathematics 2019)
https://arxiv.org/abs/1605.01418 (SISC 2017)

BIG Data


The big data opportunity
LET'S SOLVE THIS PROBLEM BY USING THE BIG DATA NONE OF US HAVE THE SLIGHTEST IDEA WHAT TO DO WITH
TOM FISH BURNE
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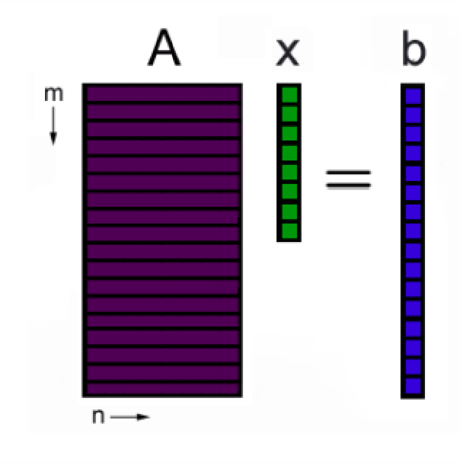
BIG Data

The Economist
Obama the warrior
Misgoverning Argentina
The economic shift from West to East
Genetically modified crops blossom
The right to eat cats and dogs
The data deluge
AND HOW TO HANDLE IT: A 14-PAGE SPECIAL REPORT

Setup

We are interested in solving highly overdetermined systems of equations (or inequalities), $A\mathbf{x} = \mathbf{b}$ ($A\mathbf{x} \leq \mathbf{b}$), where $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $m \gg n$. Rows are denoted $\mathbf{a}_i^T$.



Iterative Projection Methods

If $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$ is nonempty, these methods construct an approximation to a solution:

1. Randomized Kaczmarz Method

Applications:

1. Tomography (Algebraic Reconstruction Technique)

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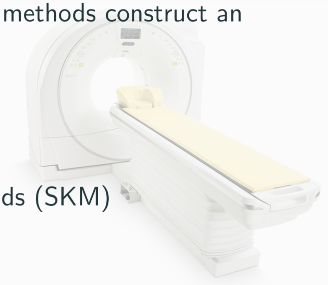
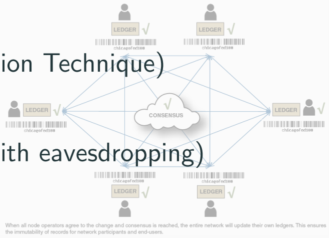
2. Motzkin's Method
3. Sampling Kaczmarz-Motzkin Methods (SKM)

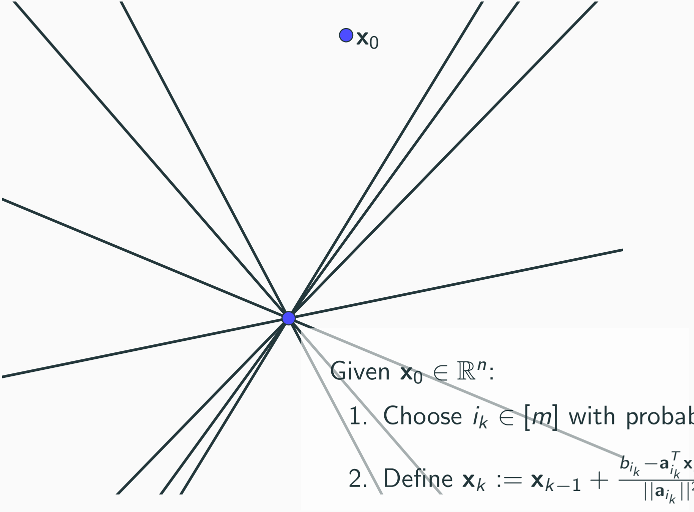


Applications:

1. Tomography (Algebraic Reconstruction Technique)
2. Linear programming
3. Average consensus (greedy gossip with eavesdropping)



Kaczmarz Method



Given $\mathbf{x}_0 \in \mathbb{R}^n$:

1. Choose $i_k \in [m]$ with probability $\frac{\|\mathbf{a}_{i_k}\|^2}{\|A\|_F^2}$.
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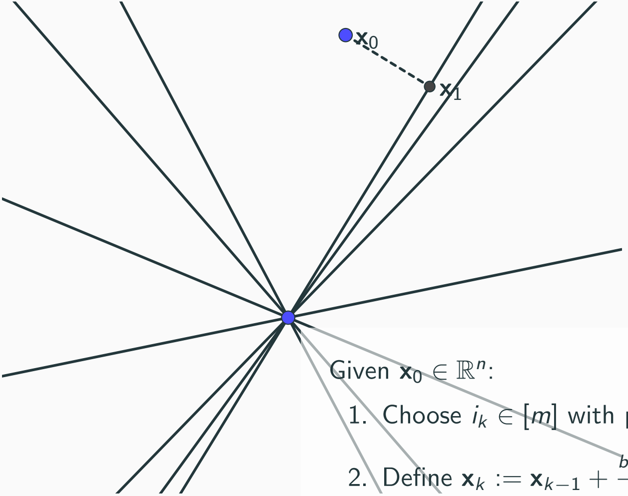

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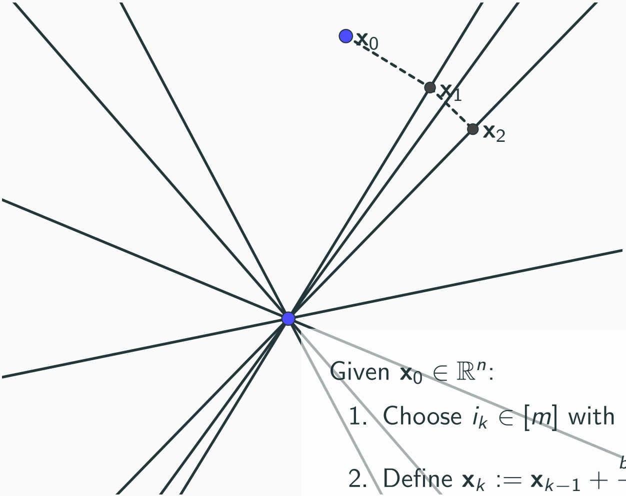

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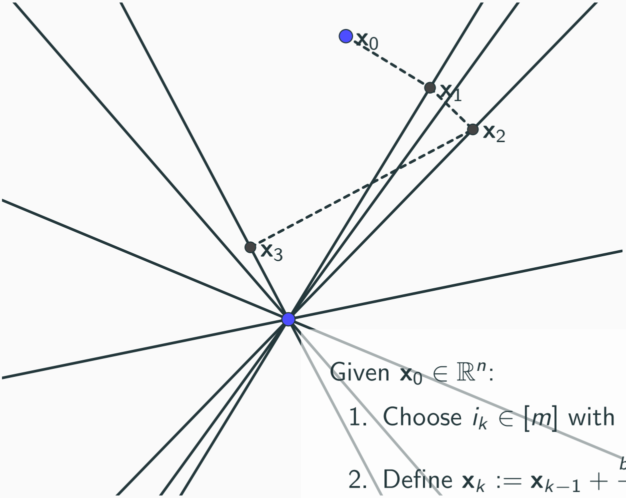

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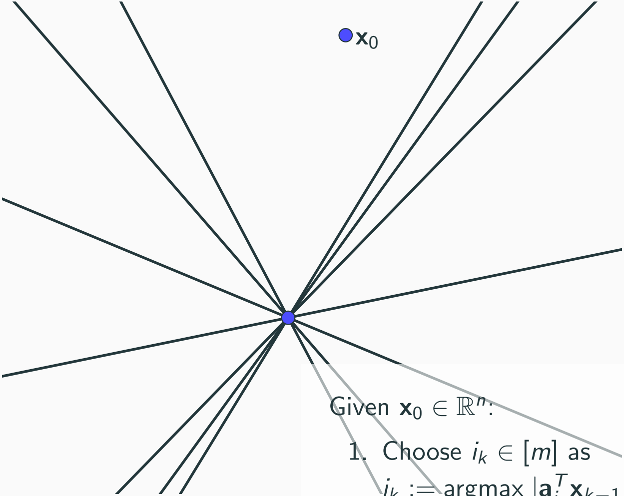

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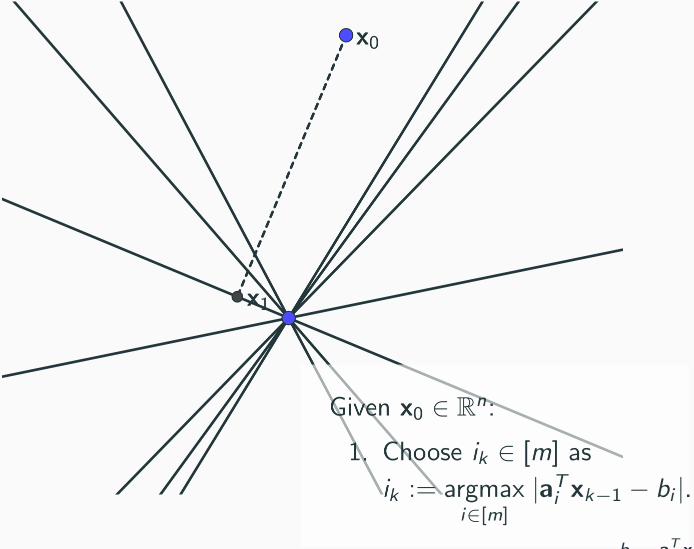

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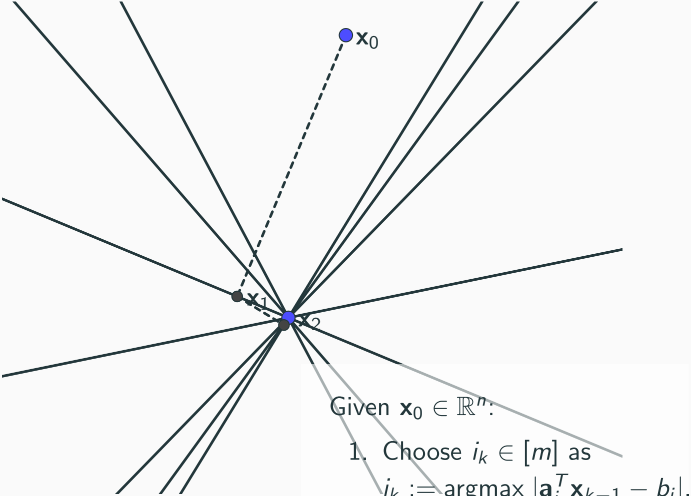

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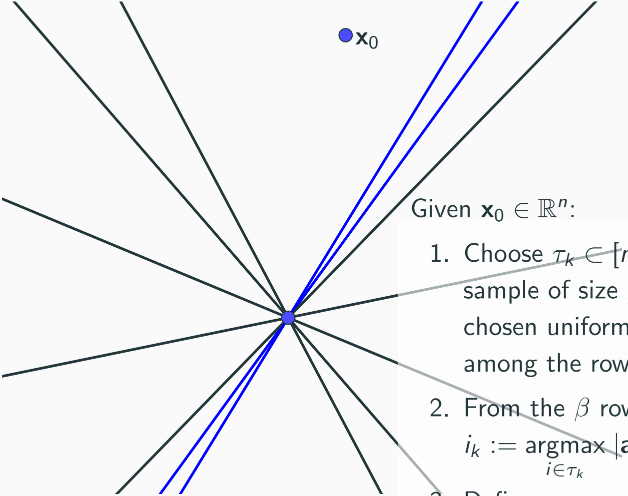
Our Hybrid Method (SKM)



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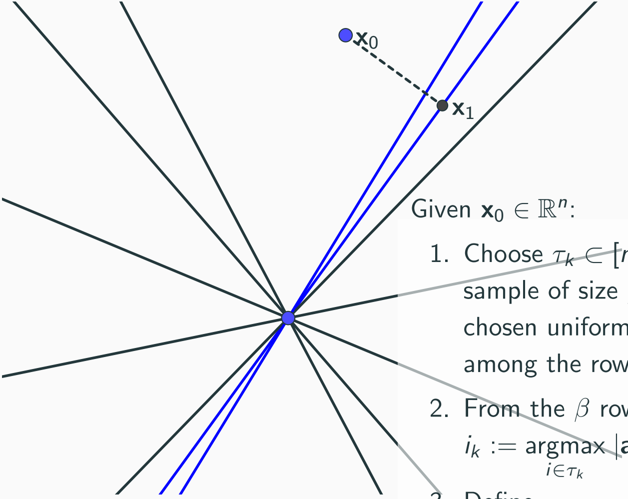

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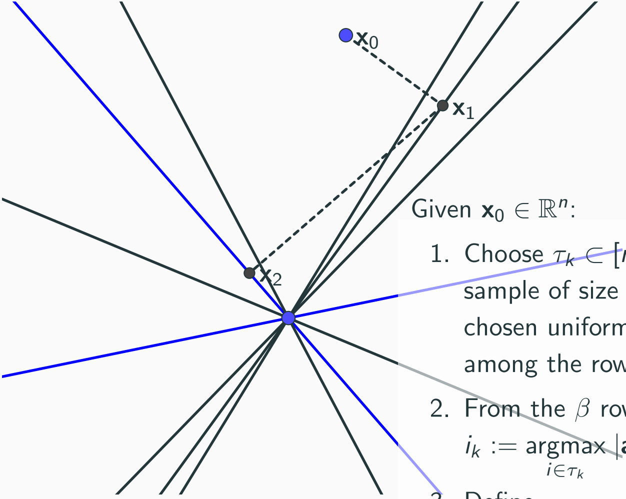

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Glimpse of HUGE Body of Literature

RK: [Strohmer-Vershynin '09], [Needell-Srebro-Ward '16]

Greedy: [Censor '81], [Nutini et al '16], [Bai-Wu '18], [Du-Gao '19]

Accel.: [Hanke-Niethammer '90], [Liu-Wright '16], [Morshed-Islam '19]

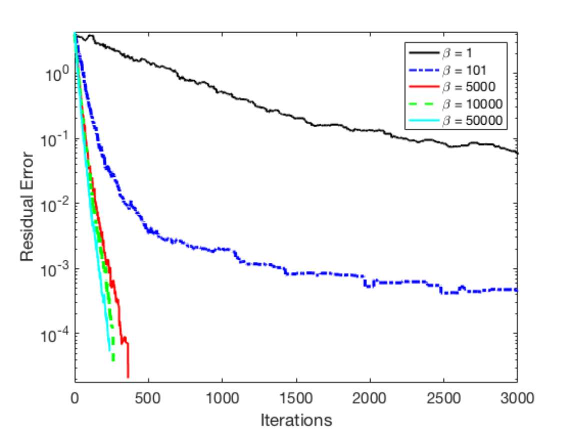
Block: [Popa et al '12], [Needell-Tropp '14], [Needell-Zhao-Zouzias '15],

Sketching: [Gower-Richtarik '15], [Needell-Rebrova '19]

Phase retrieval: [Tan-Vershynin '17], [Jeong-Güntürk '17]

LP: [Motzkin-Schoenberg '54], [Agmon '54], [Goffin '80], [Chubanov '12]

Experimental Convergence



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- A is 50000×100 Gaussian matrix, consistent system
- 'faster' convergence for larger sample size

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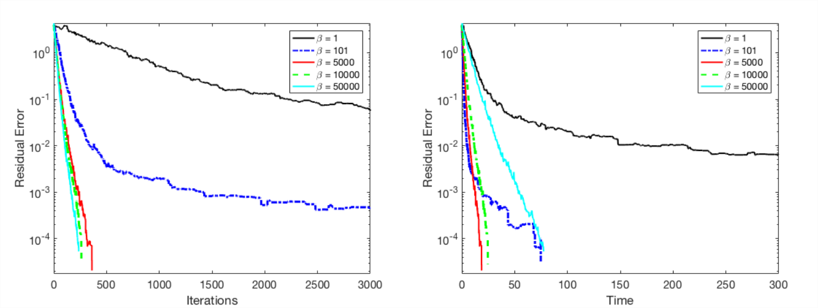

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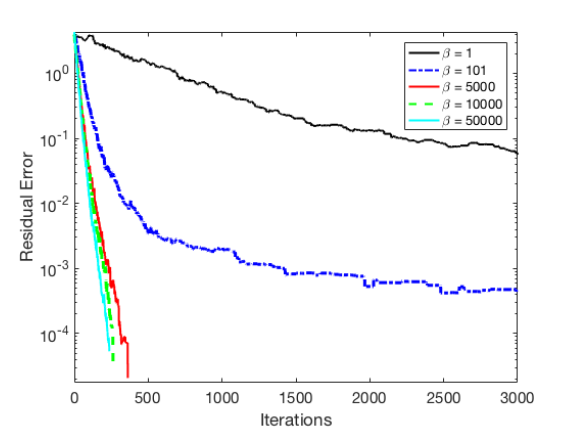

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Convergence Rates

Below are the convergence rates for the methods on a system, $A\mathbf{x} = \mathbf{b}$, which is consistent with unique solution $\mathbf{x}$, whose rows have been normalized to have unit norm.

- ▷ RK (Strohmer - Vershynin '09):

$$\mathbb{E}||\mathbf{x}_k - \mathbf{x}||_2^2 \leq \left(1 - \frac{\sigma_{\min}^2(A)}{m}\right)^k ||\mathbf{x}_0 - \mathbf{x}||_2^2$$

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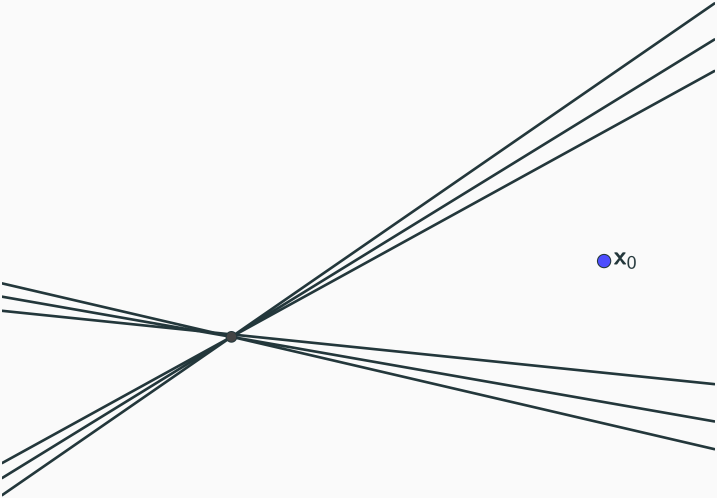
`Why are these all the same?`

A Pathological Example

Because.



Structure of the Residual

Several works have used sparsity of the residual to improve the convergence rate of greedy methods.

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However, not much sparsity can be expected in most cases. Instead, we'd like to use dynamic range of the residual to guarantee faster convergence.

$$\gamma_k := \frac{\|A\mathbf{x}_k - A\mathbf{x}\|^2}{\|A\mathbf{x}_k - A\mathbf{x}\|_\infty^2}$$

An Accelerated Convergence Rate

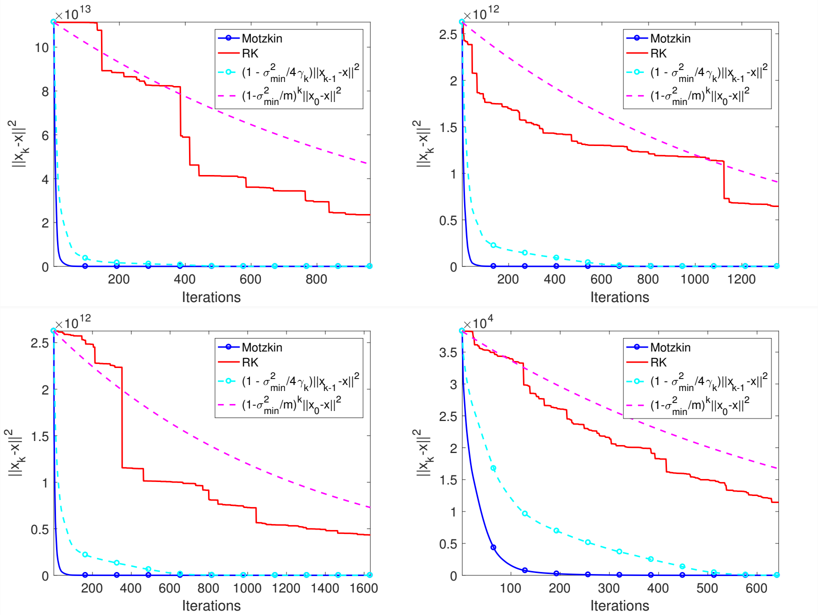
Theorem (H. - Needell '19)
Let $\mathbf{x}$ *denote the solution of the consistent, normalized system* $A\mathbf{x} = \mathbf{b}$*. Motzkin's method exhibits the (possibly highly accelerated) convergence rate:*

$$\|\mathbf{x}_T - \mathbf{x}\|^2 \leq \prod_{k=0}^{T-1} \left(1 - \frac{\sigma_{\min}^2(A)}{4\gamma_k}\right) \cdot \|\mathbf{x}_0 - \mathbf{x}\|^2$$

Here γ_k *bounds the dynamic range of the* k*th residual,* $\gamma_k := \frac{\|A\mathbf{x}_k - A\mathbf{x}\|^2}{\|A\mathbf{x}_k - A\mathbf{x}\|_\infty^2}$.

- improvement over previous result when $4\gamma_k < m$

Netlib LP Systems



Extending to SKM

Corollary (H. - Ma 2019+)
Let A be normalized so $\|\mathbf{a}_i\|_2 = 1$ for all rows $i = 1, ..., m$. If the system $A\mathbf{x} = \mathbf{b}$ is consistent with the unique solution $\mathbf{x}^*$ then the SKM method converges at least linearly in expectation and the rate depends on the dynamic range of the random sample of rows of A, τ_j. Precisely, in the $j+1$st iteration of SKM, we have

$$\mathbb{E}_{\tau_j}\|\mathbf{x}_{j+1} - \mathbf{x}^*\|_2^2 \leq \left(1 - \frac{\beta\sigma_{\min}^2(A)}{\gamma_j m}\right)\|\mathbf{x}_j - \mathbf{x}^*\|_2^2$$

where $\gamma_j = \frac{\sum_{\tau_j \in \binom{[m]}{\beta}} \|A_{\tau_j}\mathbf{x}_j - \mathbf{b}_{\tau_j}\|_2^2}{\sum_{\tau_j \in \binom{[m]}{\beta}} \|A_{\tau_j}\mathbf{x}_j - \mathbf{b}_{\tau_j}\|_\infty^2}$.

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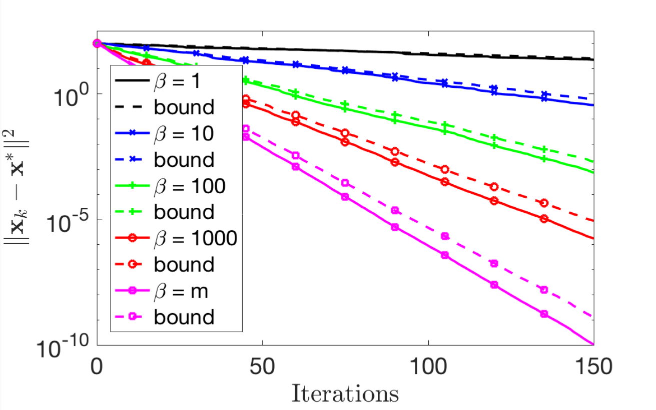

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SKM	$1 - \frac{\beta\sigma^2_{\min}(A)}{m}$		$1 - \frac{\sigma^2_{\min}(A)}{m}$	
RK	$1 - \frac{\sigma^2_{\min}(A)}{m}$			

Table 1: Contraction terms α such that $\mathbb{E}_{\tau_k}\|\mathbf{e}_k\|^2 \leq \alpha\|\mathbf{e}_{k-1}\|^2$.

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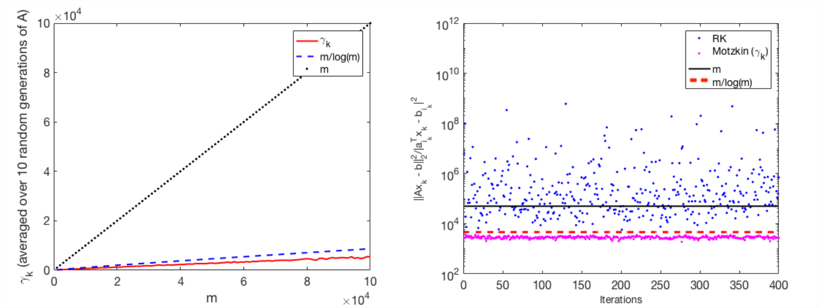
Nervous? $\gamma_k \geq \frac{\beta}{m}\sigma^2_{\min}(A)$ when A is row-normalized

γ_k: Gaussian systems



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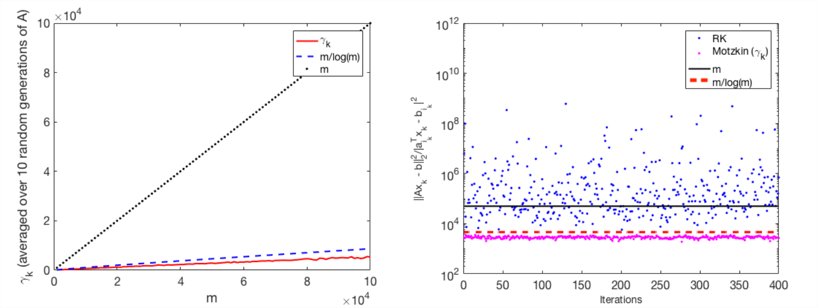
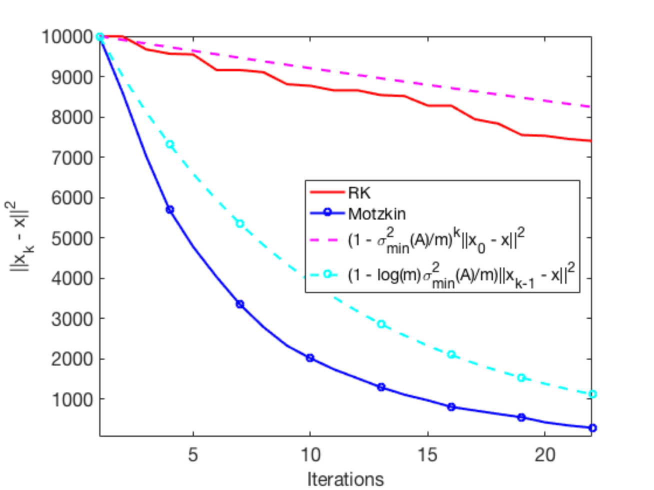


$$\gamma_k \lesssim \frac{n\beta}{\log \beta}$$

Gaussian Convergence



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proportional to norm of selected row

Generalized SKM

Given $\mathbf{x}_0 \in \mathbb{R}^n$:

1. Choose $\tau_k \in \binom{[m]}{\beta_k}$ according to $p_{\mathbf{x}_{k-1}}$.
2. Choose $i_k := t(\tau_k, \mathbf{x}_{k-1})$.
3. Define $\mathbf{x}_k := \mathbf{x}_{k-1} + \frac{b_{i_k} - \mathbf{a}_{i_k}^T \mathbf{x}_{k-1}}{||\mathbf{a}_{i_k}||^2} \mathbf{a}_{i_k}$.
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Generalized Result

Theorem (H. - Ma 2019+)

Let $\mathbf{x}^*$ denote the unique solution to the system of equations $A\mathbf{x} = \mathbf{b}$. Then generalized SKM converges at least linearly in expectation and the bound on the rate depends on the dynamic range, γ_k of the random sample of β_k rows of A, τ_k. Precisely, in the kth iteration of generalized SKM, we have

$$\mathbb{E}_{\tau_k}\|\mathbf{x}_k - \mathbf{x}^*\|^2 \leq \left(1 - \frac{\beta_k \binom{m}{\beta_k} \sigma^2_{\min}(A)}{\gamma_k m \sum_{\tau \in \binom{[m]}{\beta_k}} \|\mathbf{a}_{t(\tau, \mathbf{x}_{k-1})}\|^2}\right) \|\mathbf{x}_{k-1} - \mathbf{x}^*\|^2.$$

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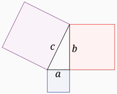
$\triangleright$ If all rows of A have the same norm, then

$$\mathbb{E}_{\tau_k}\|\mathbf{x}_k - \mathbf{x}^*\|^2 \leq \left(1 - \frac{\beta_k \sigma_{\min}^2(A)}{\gamma_k \|A\|_F^2}\right) \|\mathbf{x}_{k-1} - \mathbf{x}^*\|^2.$$

Sketch of Proof

$$\left\|\mathbf{x}_k - \mathbf{x}^*\right\|^2 = \left\|\mathbf{x}_{k-1} - \mathbf{x}^*\right\|^2 - \frac{\left\|A_{\tau_k}\mathbf{x}_{k-1} - \mathbf{b}_{\tau_k}\right\|_\infty^2}{\left\|\mathbf{a}_{t(\tau_k, \mathbf{x}_{k-1})}\right\|^2}$$



[1]

[1] Originally created by en:User:Michael Hardy, then scaled, with colour and labels being added by en:User:Wapcaplet, transformed in svg format by fr:Utilisateur:Steff, changed colors and font by de:Leo2004. (https://commons.wikimedia.org/wiki/File:Pythagorean.svg)

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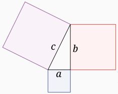

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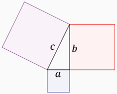

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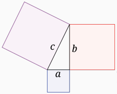

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$$\begin{aligned}
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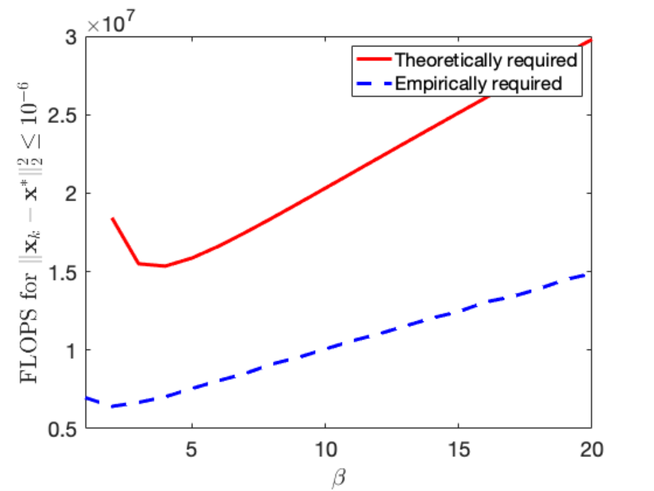

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Thanks for listening!

Questions?

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