3.26pt

Randomized Projections for Corrupted Linear Systems

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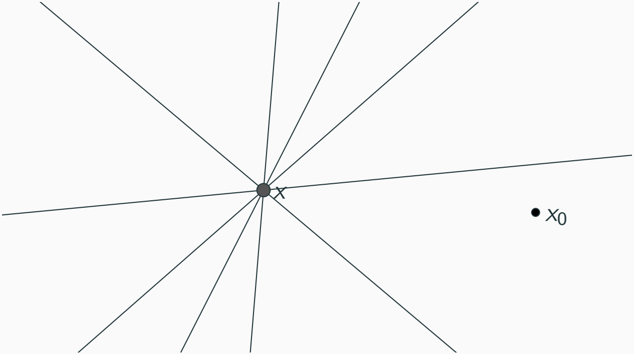
~~**(Noisy)**~~	**Error (e):**	small, evenly distributed entries
	Solution ($x_{\mathbf{LS}}$):	$x_{\text{LS}} \in \text{argmin}\|Ax - b - e\|^2$

Why not least-squares?

Randomized Kaczmarz

RK

1. *Start with initial guess* x_0
2. $x_{k+1} = x_k + \frac{b_{i_k} - a_{i_k}^T x_k}{\|a_{i_k}\|^2} a_{i_k}$ *where* $i_k \in [m]$ *is chosen randomly*
3. *Repeat (2)*



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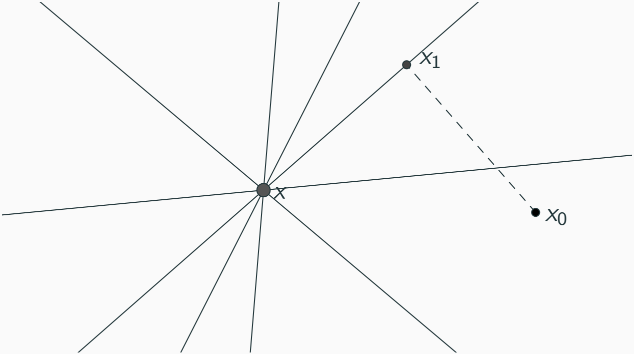

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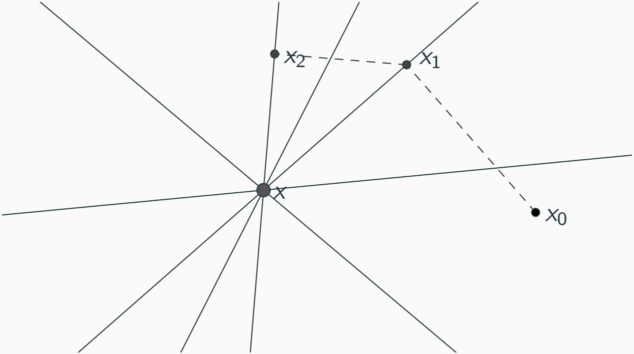

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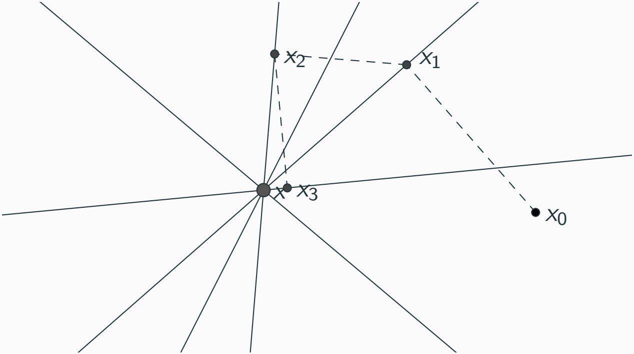

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Theorem (Strohmer-Vershynin, 2008)

If $Ax = b$ *is consistent and RK is used with* $\mathbb{P}[i_k = j] = \|a_j\|^2/\|A\|_F^2$ *then iterates converge linearly in expectation with*

$$\mathbb{E}\|x_k - x\|^2 \leq \left(1 - \frac{1}{\|A\|_F^2 \|A^{-1}\|^2}\right)^k \|x_0 - x\|^2.$$

Proposed Method

Goal: Use RK to detect the corrupted equations with high probability.

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Lemma (H.-Needell)

Let $\epsilon^* = \min_{i \in [m]} |Ax^* - b|_i = |e_i|$ *and suppose* $|supp(e)| = s$*. If* $||a_i|| = 1$ *for* $i \in [m]$ *and* $||x - x^*|| < \frac{1}{2}\epsilon^*$ *we have that the* $d \leq s$ *indices of largest magnitude residual entries are contained in* $supp(e)$*. That is, we have* $D \subset supp(e)$*, where*

$$D = \underset{D \subset [A], |D| = d}{argmax} \sum_{i \in D} |Ax - b|_i.$$

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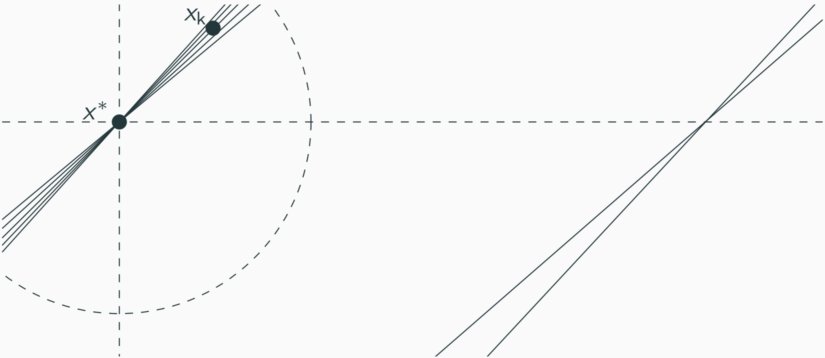

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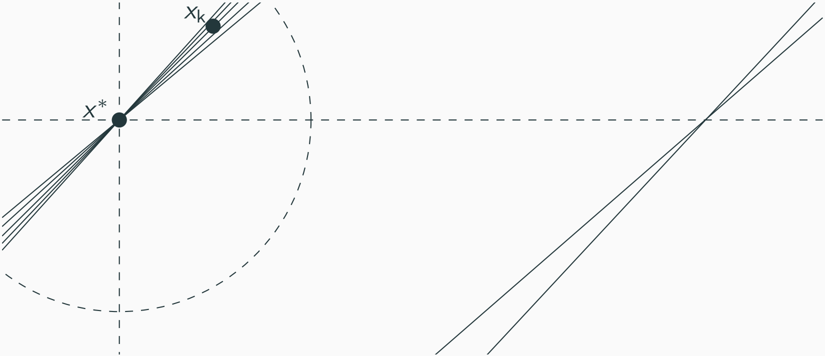


We call $\epsilon^*/2$ the *detection horizon*.

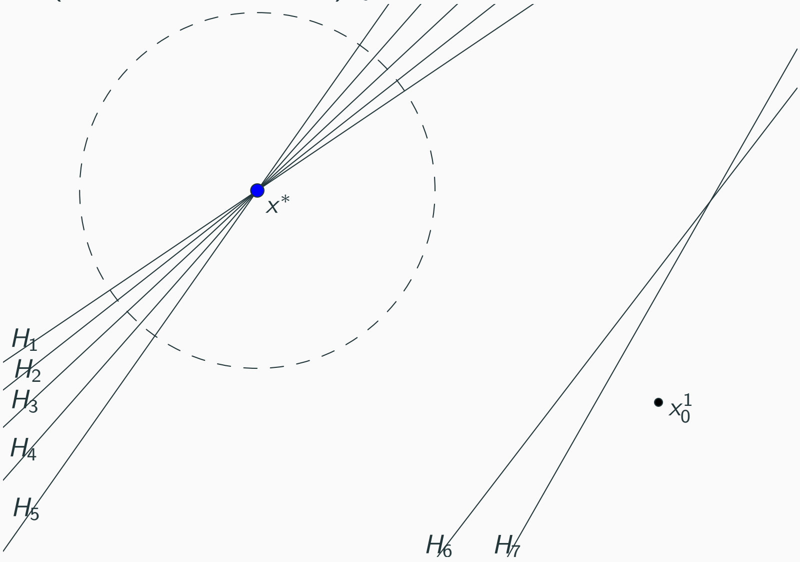
Proposed Method

Method 1 Windowed Kaczmarz

1: **procedure** $\mathrm{WK}(A, b, k, W, d)$
2: $S = \emptyset$
3: **for** $i = 1, 2, \ldots W$ **do**
4: $x_k^i = k$th iterate produced by RK with $x_0 = 0$, A, b.
5: $D = d$ indices of the largest entries of the residual, $|Ax_k^i - b|$.
6: $S = S \cup D$
7: **return** x, where $A_{S^C} x = b_{S^C}$

Example

WK(A**,**b**,**$k = 2$**,**$W = 3$**,**$d = 1$**):** $j = 1,\ i = 1,\ S = \emptyset$



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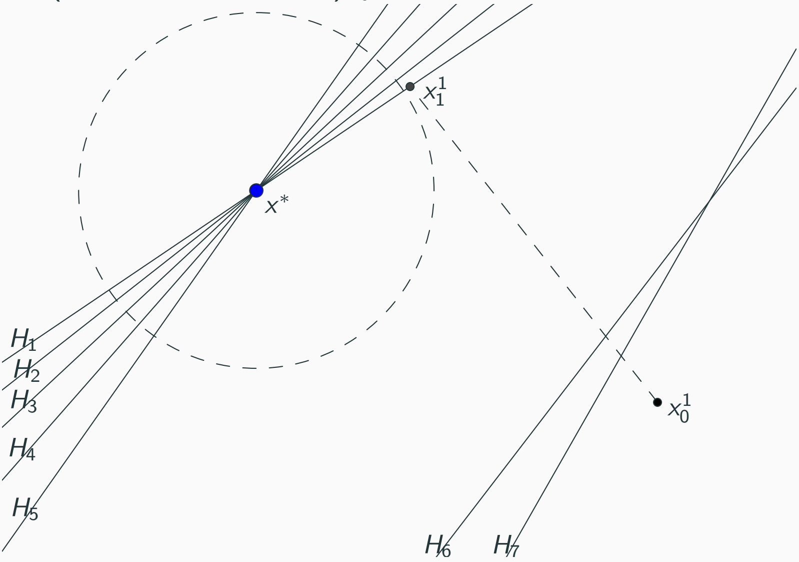
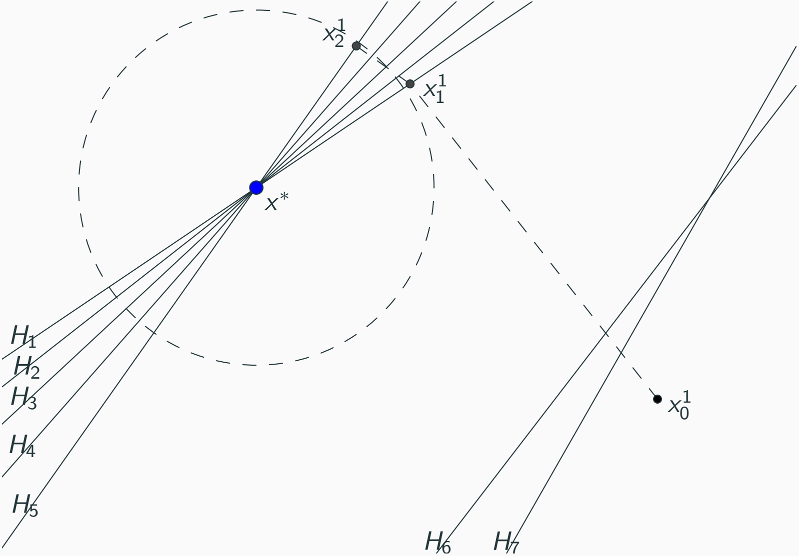


Example

WK(A,b,$k=2$,$W=3$,$d=1$): $j=2$, $i=1$, $S=\{7\}$



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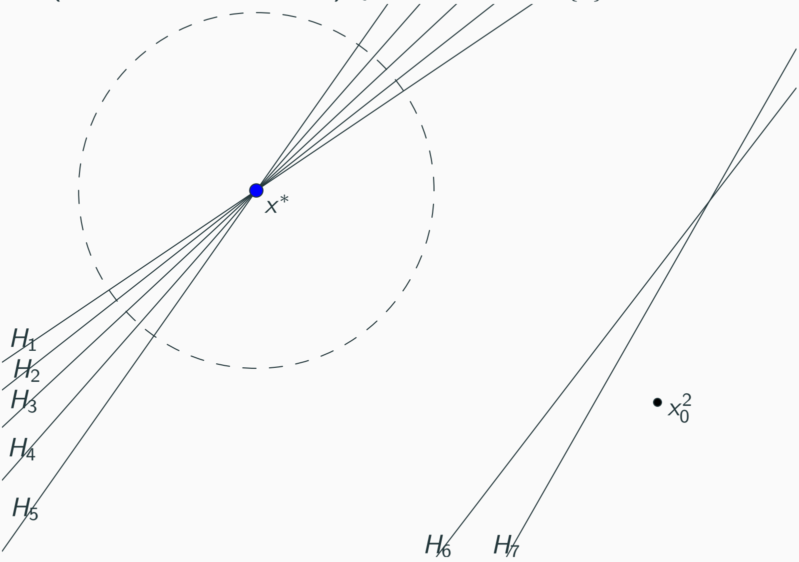

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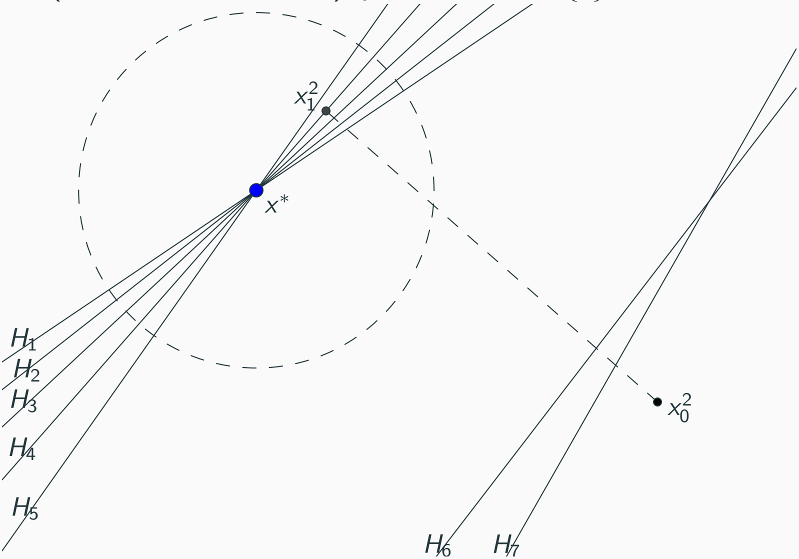

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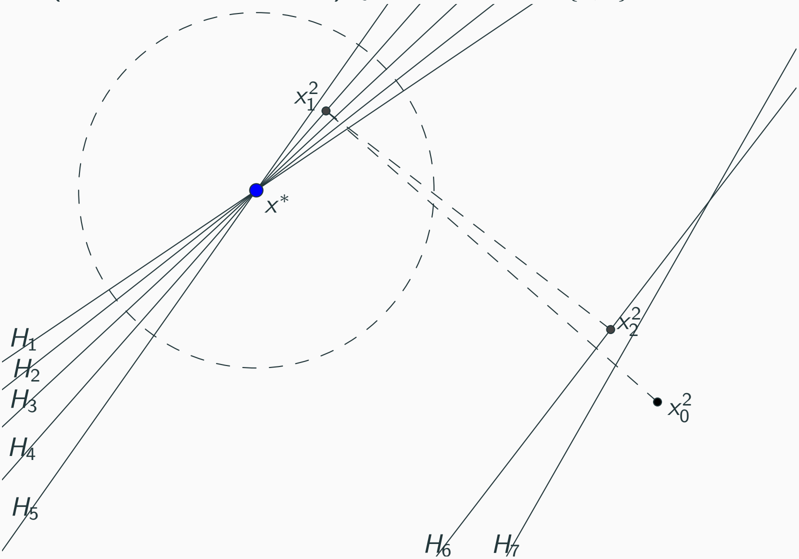
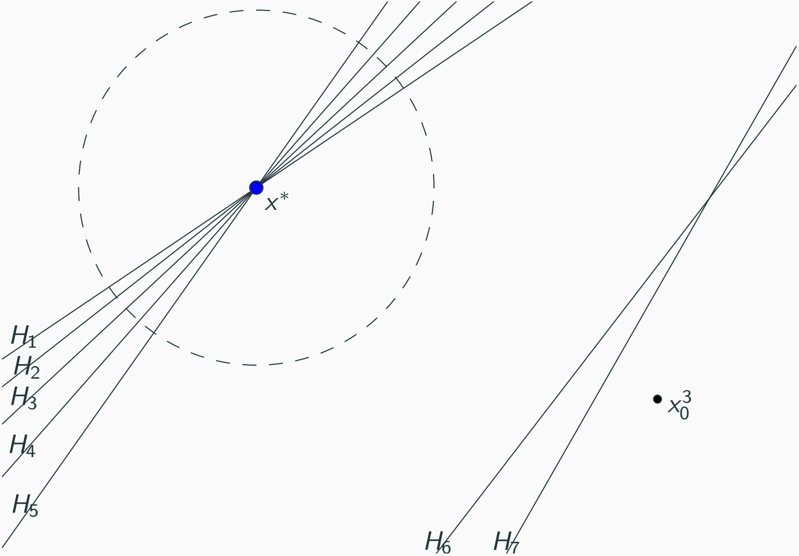
WK(A,b,$k=2$,$W=3$,$d=1$): $j=2,\ i=2,\ S=\{7,5\}$



Example

WK(A**,**b**,**$k = 2$**,**$W = 3$**,**$d = 1$**):** $j = 1,\ i = 3,\ S = \{7, 5\}$



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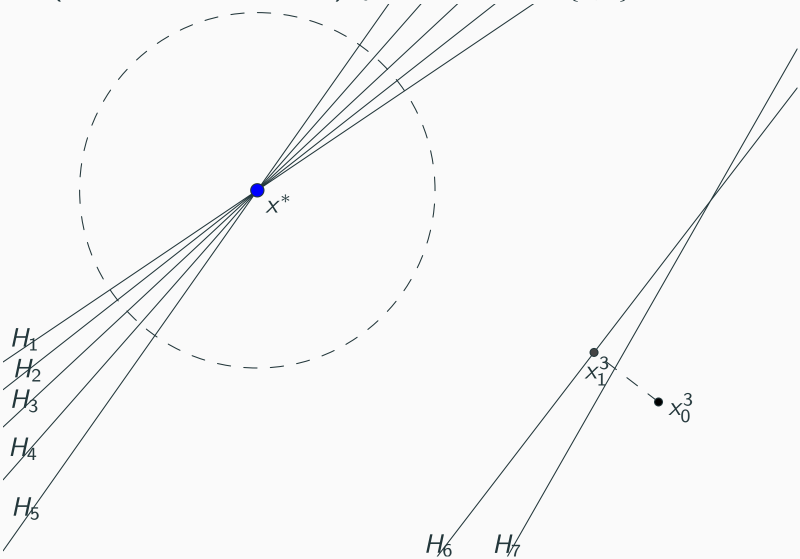
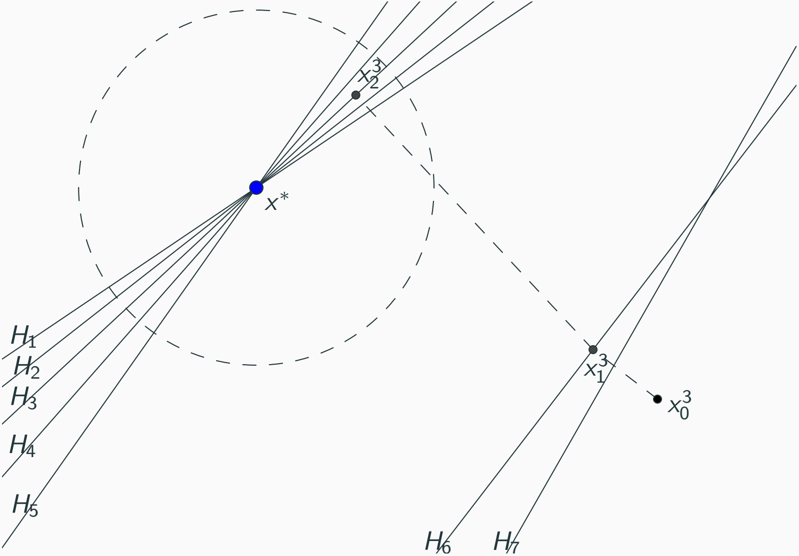


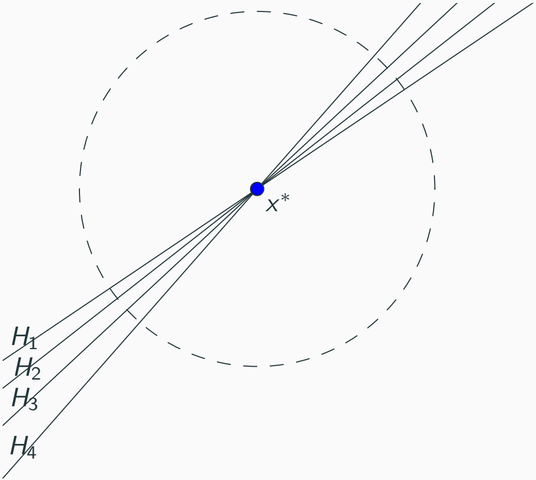
Example

WK(A**,**b**,**$k = 2$**,**$W = 3$**,**$d = 1$**):** $j = 2,\ i = 3,\ S = \{7, 5, 6\}$



Example

Solve $A_{S^c} x = b_{S^c}$.



Theoretical Guarantees

Lemma (H.-Needell)

Let $\epsilon^* = \min_{i\in[m]} |Ax^* - b|_i = |e_i|$ *and suppose* $|supp(e)| = s$. *Assume that* $||a_i|| = 1$ *for all* $i \in [m]$ *and let* $0 < \delta < 1$. *Define*

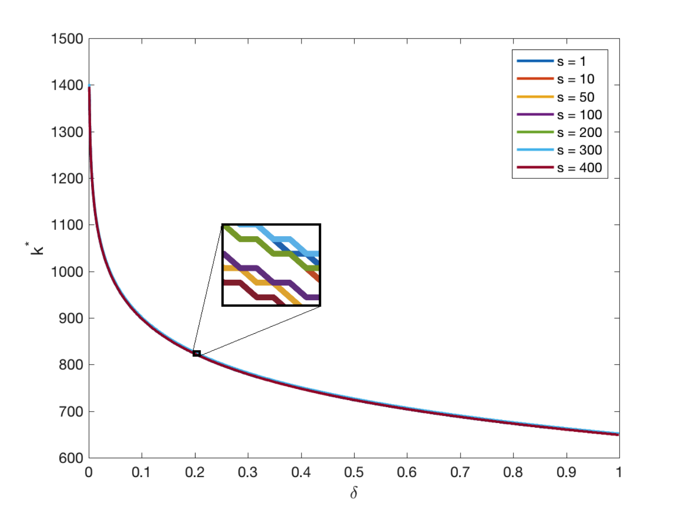
$$k^* = \left\lceil \frac{\log\left(\frac{\delta(\epsilon^*)^2}{4||x^*||^2}\right)}{\log\left(1 - \frac{\sigma^2_{\min}(A_{supp(e)^C})}{m-s}\right)} \right\rceil.$$

Then in window i *of the Windowed Kaczmarz method, the iterate produced by the RK iterations,* $x^i_{k^*}$ *satisfies*

$$\mathbb{P}\Big[||x^i_{k^*} - x^*|| \leq \frac{1}{2}\epsilon^*\Big] \geq p := (1-\delta)\Big(\frac{m-s}{m}\Big)^{k^*}.$$

Theoretical Guarantee Values (Gaussian $A \in \mathbb{R}^{50000\times 100}$)

$$k^* = \left\lceil \frac{\log\left(\frac{\delta(\epsilon^*)^2}{4||x^*||^2}\right)}{\log\left(1-\frac{\sigma_{\min}^2(A_{\text{supp}(e)^C})}{m-s}\right)} \right\rceil$$



Theoretical Guarantee Values (Correlated $A \in \mathbb{R}^{50000\times 100}$)

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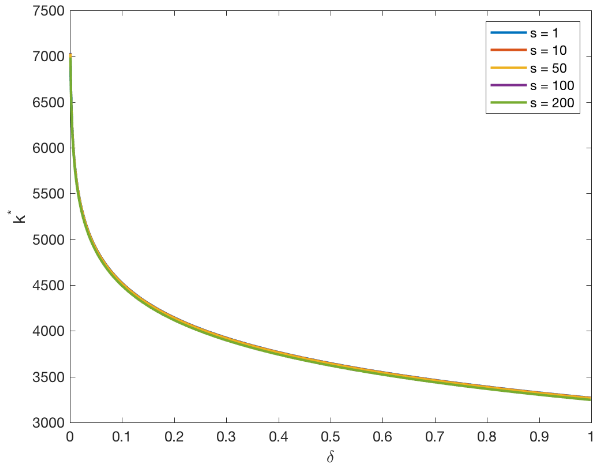

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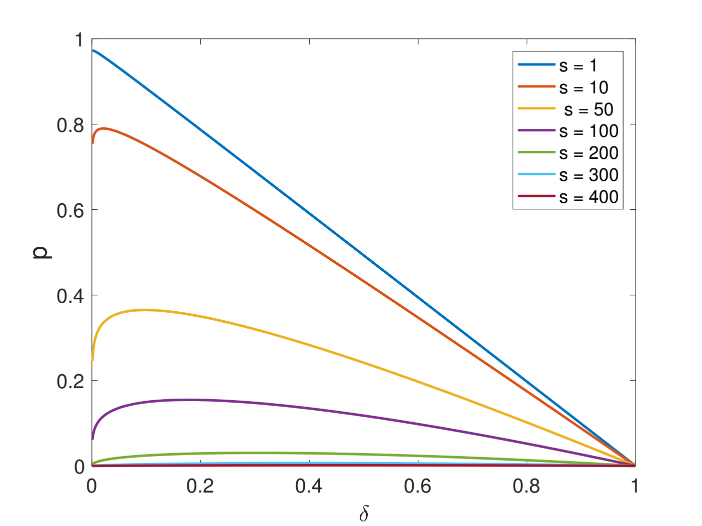

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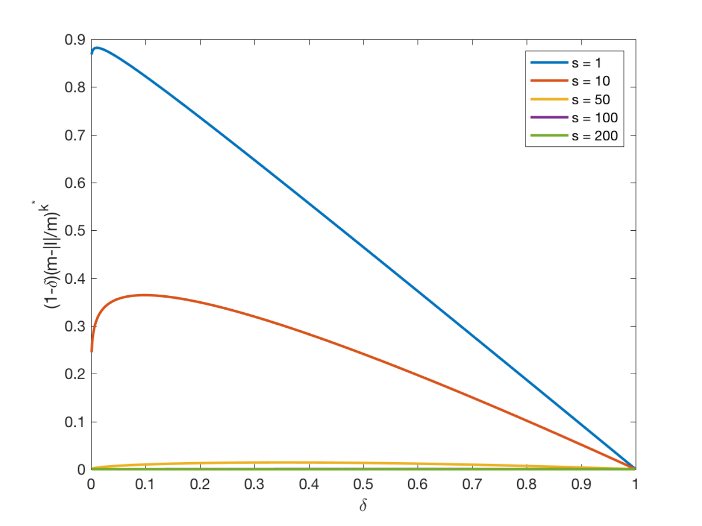

Theoretical Guarantees

Theorem (H.-Needell)

Assume that $||a_i|| = 1$ *for all* $i \in [m]$ *and let* $0 < \delta < 1$*. Suppose* $d \geq s = |supp(e)|$, $W \leq \lfloor \frac{m-n}{d} \rfloor$ *and* k^* *is as given in lemma 2. Then the Windowed Kaczmarz method on* A, b *will detect the corrupted equations (*$supp(e) \subset S$*) and the remaining equations given by* $A_{[m]-S}, b_{[m]-S}$ *will have solution* x^* *with probability at least*

$$p_W := 1 - \left[1 - (1-\delta)\left(\frac{m-s}{m}\right)^{k^*}\right]^W.$$

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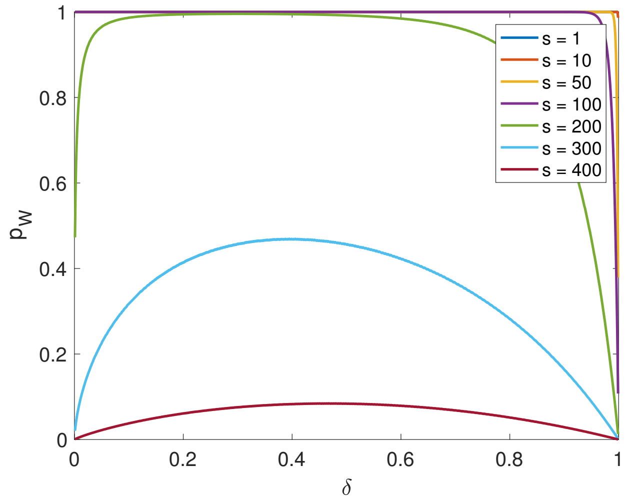
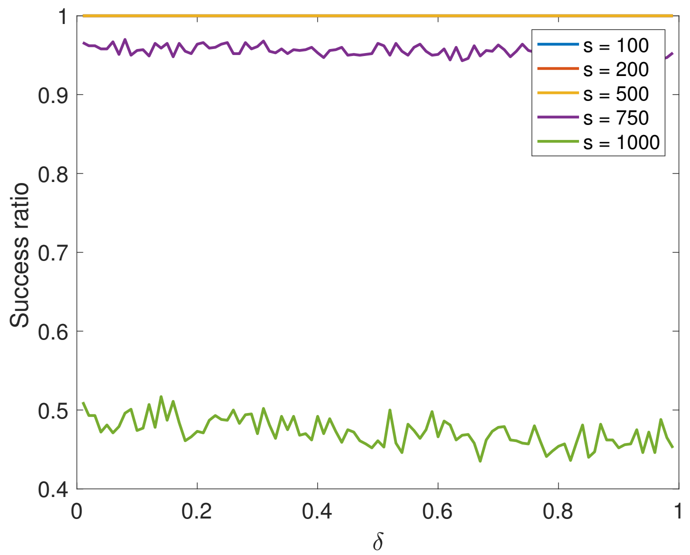


Experimental Values (Gaussian $A \in \mathbb{R}^{50000\times 100}$)



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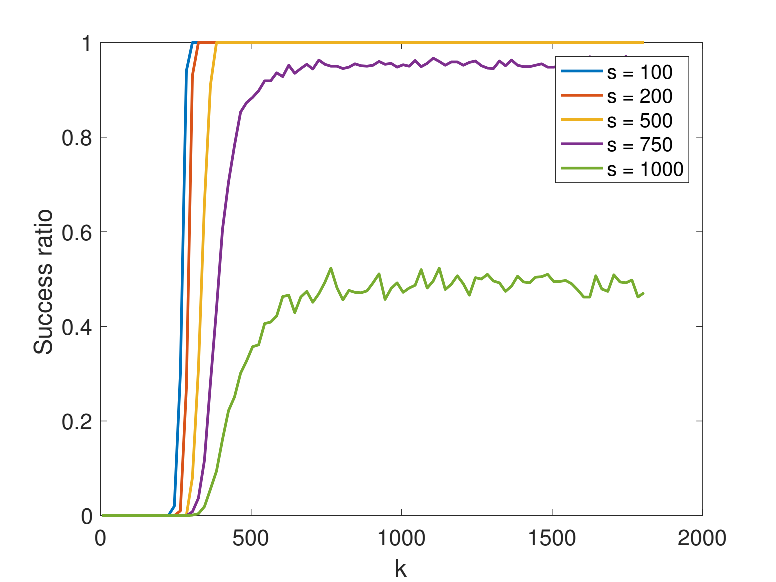

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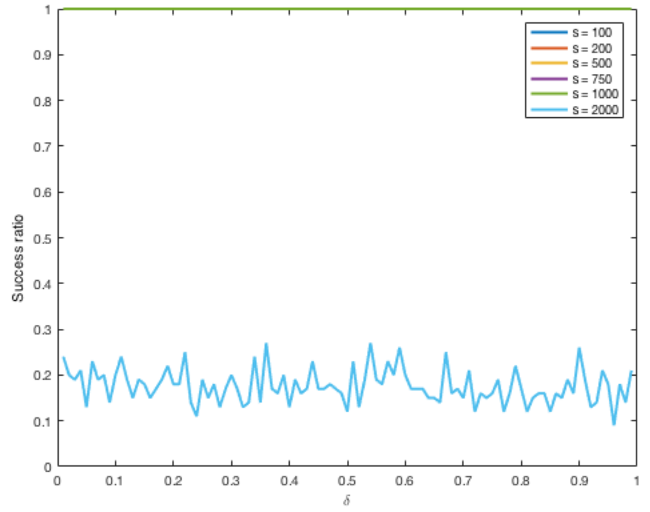

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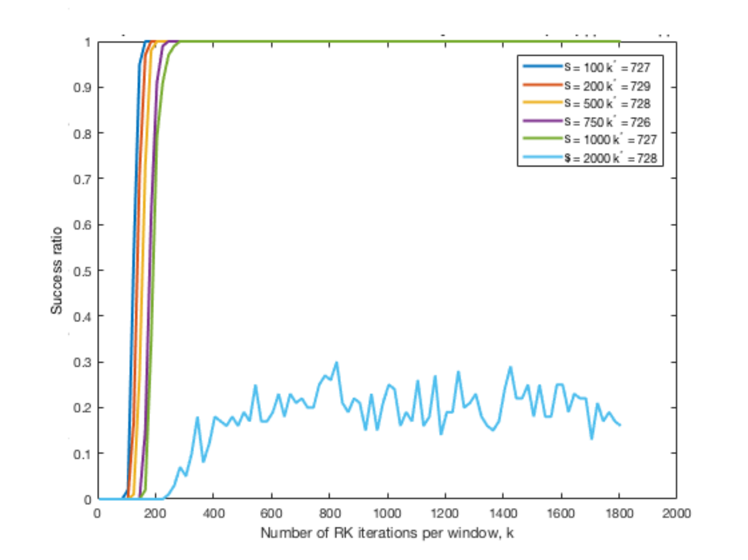

Theoretical Guarantee Values (Correlated $A \in \mathbb{R}^{50000\times 100}$)

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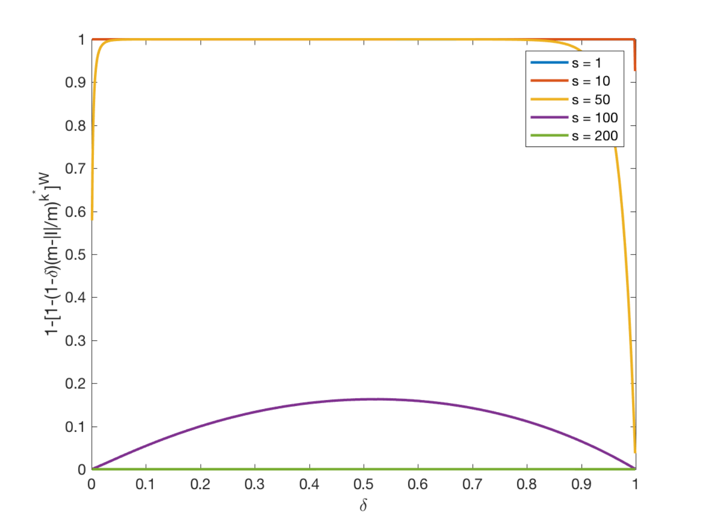

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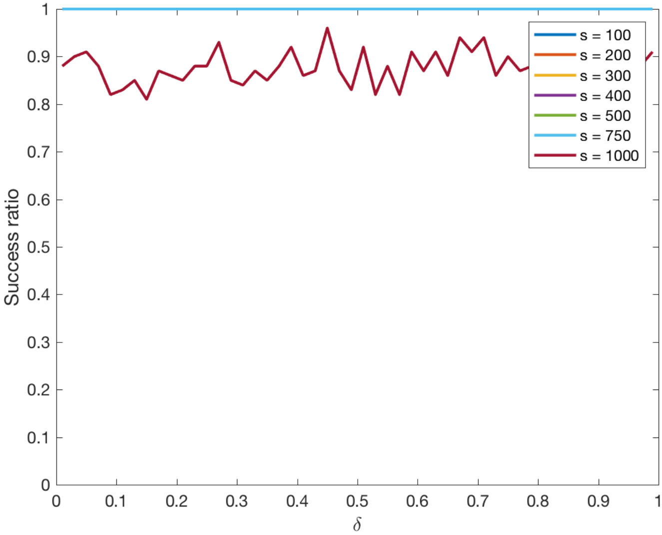

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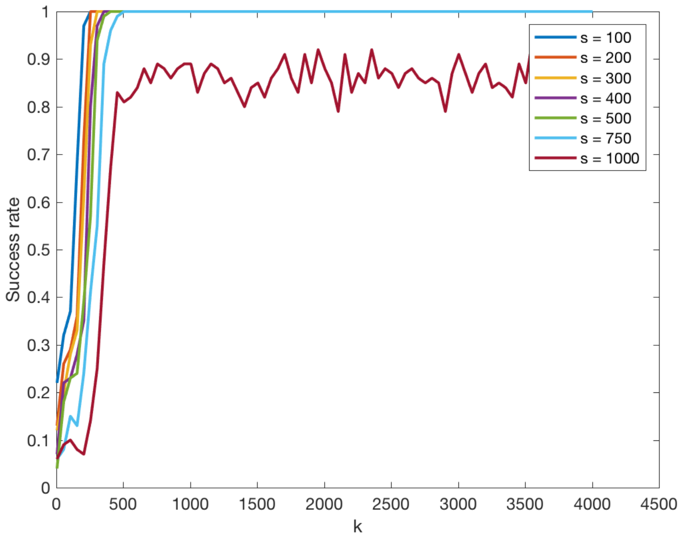

Conclusions and Future Work

- randomized projection methods are able to detect corruption
- often experimental results far outperform theoretical guarantees

- performance on real data
- reduce dependence on artificial parameters

The End

Thanks! Questions?

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