Wolfe's Combinatorial Method is Exponential

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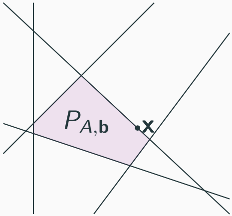
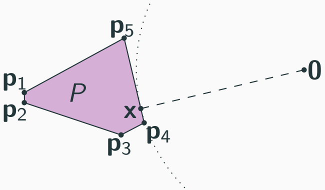
joint with Jesús De Loera and Luis Rademacher
https://arxiv.org/abs/1710.02608

Projection Algorithms for Convex and Combinatorial Optimization

Two Problems

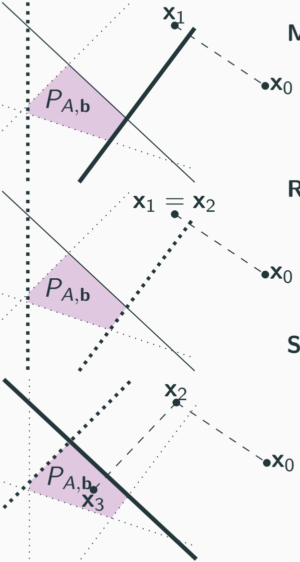
Linear Feasibility (LF)**:** Given a rational matrix A and a rational vector $\mathbf{b}$, if $P_{A,\mathbf{b}} := \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$ is nonempty, output a rational $\mathbf{x} \in P_{A,\mathbf{b}}$, otherwise output NO.





Minimum Norm Point (MNP)**:** Given rational points $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_m} \in \mathbb{R}^n$ defining $P := \text{conv}(\mathbf{p_1}, \mathbf{p_2}, ..., \mathbf{p_m})$, output rational $\mathbf{x} = \text{argmin}_{\mathbf{q} \in P} \|\mathbf{q}\|^2$.

Iterative Projection Methods for LF



Motzkin's Method (MM)

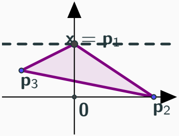
- *On Motzkin's Method for Inconsistent Linear Systems* (joint with D. Needell)
  `https://arxiv.org/abs/1802.03126`

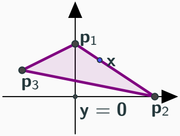
Randomized Kaczmarz (RK) Method

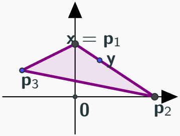
- *Randomized Projection Methods for Corrupted Linear Systems* (joint with D. Needell)
  `https://arxiv.org/abs/1803.08114`

Sampling Kaczmarz-Motzkin (SKM) Methods

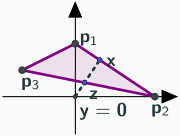
- *A Sampling Kaczmarz-Motzkin Algorithm for Linear Feasibility* (joint with J. A. De Loera and D. Needell)
  SIAM Journal on Scientific Computing, 2017
  `https://arxiv.org/abs/1605.01418`

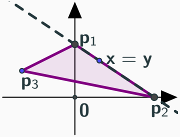


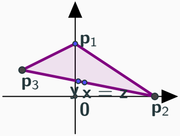




- *The Minimum Euclidean-Norm Point on a Convex Polytope: Wolfe's Combinatorial Algorithm is Exponential* (joint J. A. De Loera and L. Rademacher) STOC, 2018 `https://arxiv.org/abs/1710.02608`







Applications and Connections

`LF`:

- ▷ linear programming
- ▷ support vector machine
- ▷ linear equations

`MNP`:

- ▷ submodular function minimization
- ▷ colorful linear programming

Theorem (De Loera, H., Rademacher '17)
`LF` reduces to `MNP` on a simplex in strongly-polynomial time.

Minimum Norm Point (MNP(P))

Minimum Norm Point in Polytope

We are interested in solving the problem (MNP(P)):

$$\min_{\mathbf{x} \in P} \|\mathbf{x}\|_2$$

where P is a polytope, and determining the minimum dimension face, F, which achieves distance $\|\mathbf{x}\|_2$.

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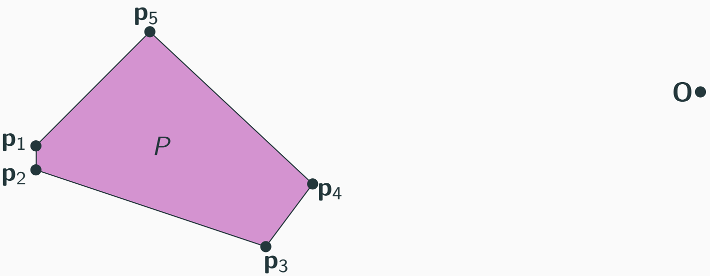
where P is a polytope, and determining the minimum dimension face, F, which achieves distance $\|\mathbf{x}\|_2$.

Reminder: A *polytope*, P, is the convex hull of points $\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m$,

$$P = \left\{ \sum_{i=1}^{m} \lambda_i \mathbf{p}_i : \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0 \text{ for all } i = 1, 2, ..., m \right\}.$$

Minimum Norm Point in Polytope



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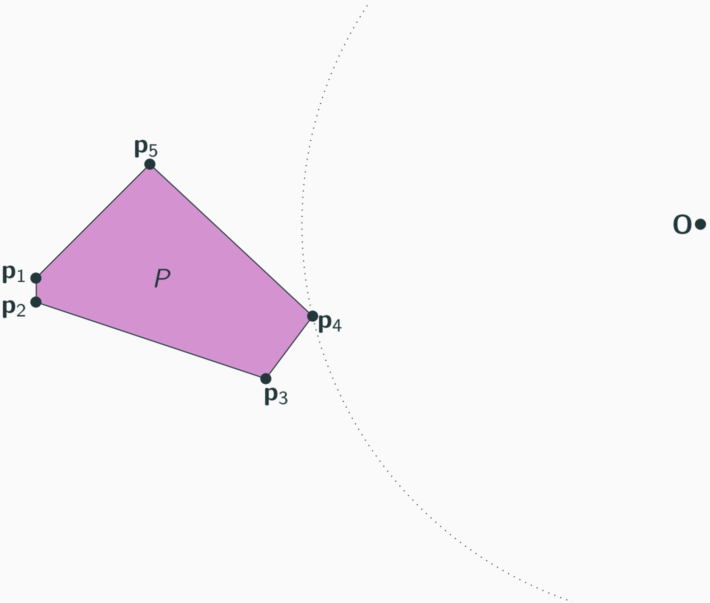

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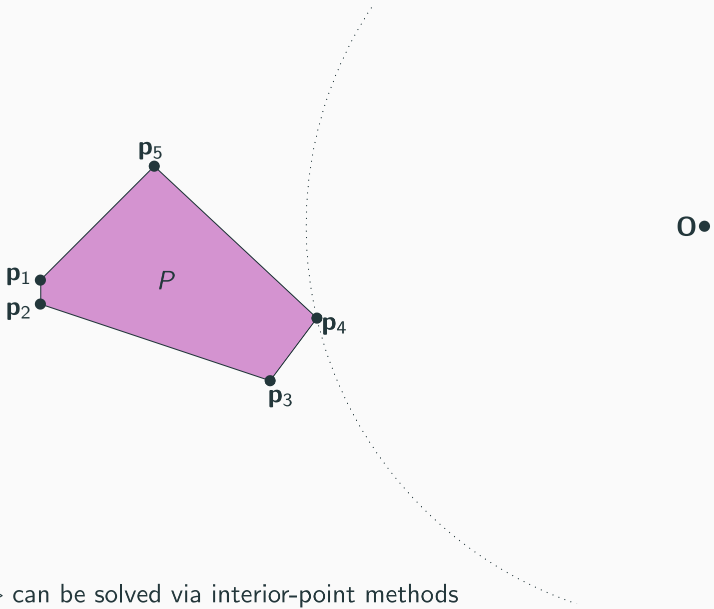

▷ can be solved via interior-point methods

Applications

- arbitrary polytope projection

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- nearest point problem for transportation polytopes

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- nearest point problem for transportation polytopes
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- machine learning - vision, large-scale learning
- compute distance to polytope

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If a strongly-polynomial method for projection onto a polytope exists then this gives a strongly-polynomial method for LP.

It was previously known that linear programming reduces to MNP on a polytope in weakly-polynomial time [Fujishige, Hayashi, Isotani '06].

Spoiler

Theorem (De Loera, H., Rademacher '17)

There exists a family of polytopes on which Wolfe's method requires exponential time to compute the MNP.

Wolfe's Optimality Condition

Theorem (Wolfe '74)

Let $P = conv(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m)$. *Then* $\mathbf{x} \in P$ *is* $MNP(P)$ *if and only if*

$$\mathbf{x}^T \mathbf{p}_j \geq \|\mathbf{x}\|_2^2 \text{ for all } j = 1, 2, ..., m.$$

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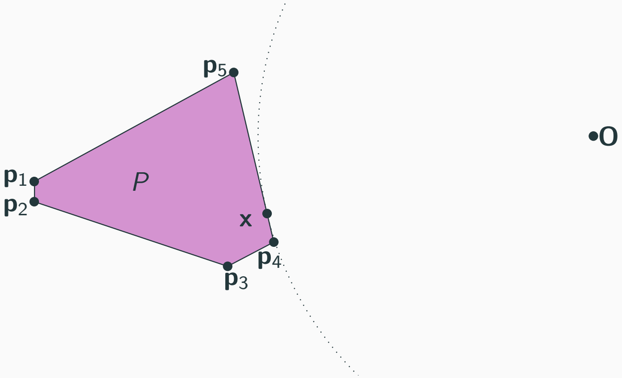

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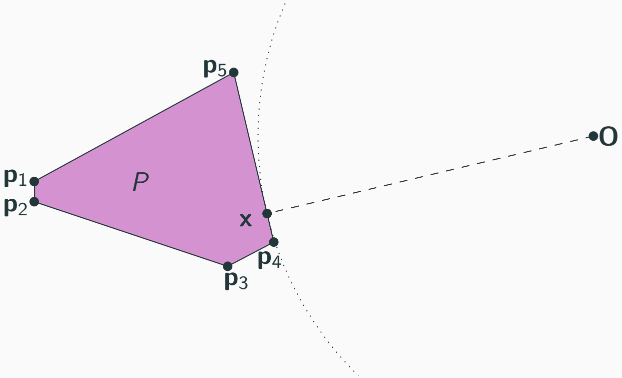

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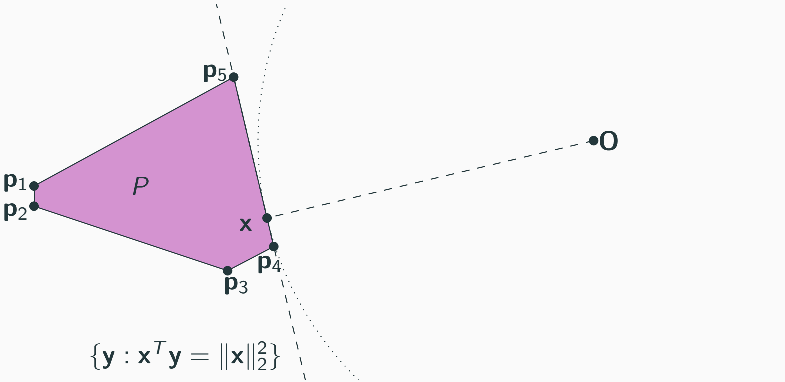
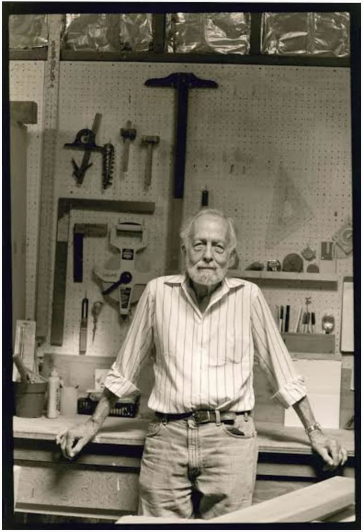


Wolfe's Method

Philip Wolfe



- Frank-Wolfe method
- Dantzig-Wolfe decomposition
- simplex method for quadratic programming

Intuition and Definitions

Idea: Exploit linear information about the problem in order to progress towards the quadratic solution.

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Def: An affinely independent set of points $Q = \{\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_k\}$ is a *corral* if $\text{MNP}(\text{aff}(Q)) \in \text{relint}(\text{conv}(Q))$.

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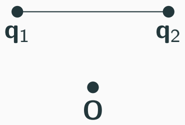

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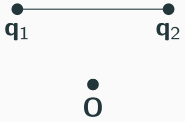

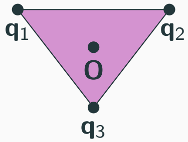

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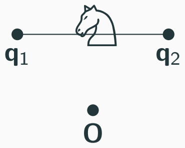

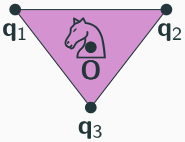

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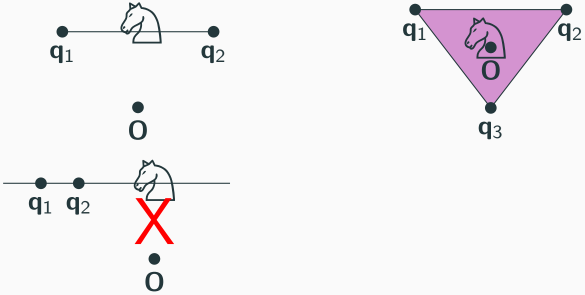

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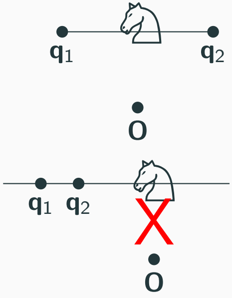

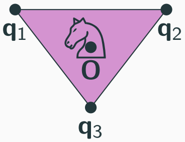

Note: Singletons are corrals.

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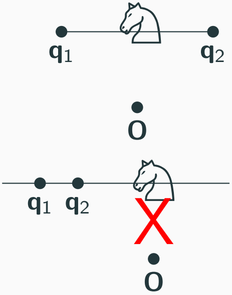

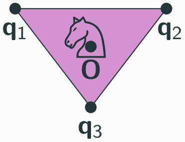

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Note: There is a corral in P whose convex hull contains $\text{MNP}(P)$.

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Wolfe's method : combinatorial method for computing projection onto a vertex-representation polytope (any dimension, any number of points)

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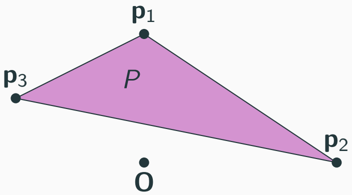
Wolfe's method : combinatorial method for computing projection onto a vertex-representation polytope (any dimension, any number of points)

- pivots between corrals which may contain $\mathrm{MNP}(P)$
- projects onto affine hull of sets to check whether a corral
- optimality criterion checks if correct corral

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
 $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
 $C = C \cup \{\mathbf{p}_j\}$
 $\mathbf{y} = \text{MNP}(\text{aff}(C))$
 while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
  $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
  $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
  $\mathbf{x} = \mathbf{z}$
  $\mathbf{y} = \text{MNP}(\text{aff}(C))$
 $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$



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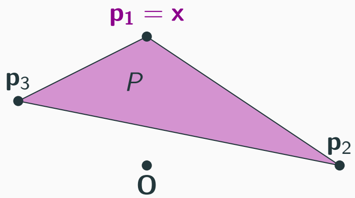

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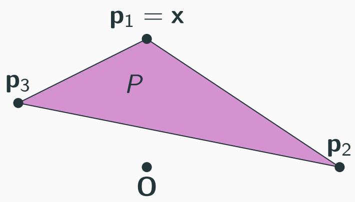

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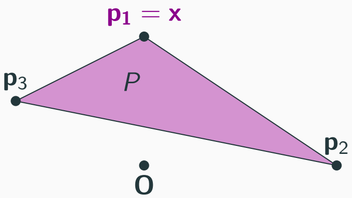

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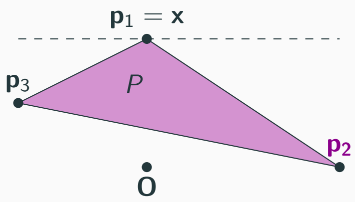

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  $\mathbf{y} = \text{MNP}(\text{aff}(C))$

  while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$

    $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$

    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)

    $\mathbf{x} = \mathbf{z}$

    $\mathbf{y} = \text{MNP}(\text{aff}(C))$

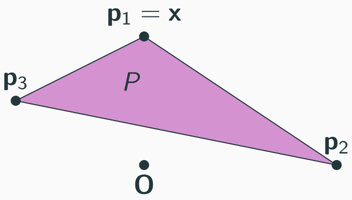
  $\mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$

$\mathbf{p}_2 = (3, 0)$

$\mathbf{p}_3 = (-2, 1)$



$\mathbf{C} = \{\mathbf{p_1}, \mathbf{p_2}\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$

$C = \{\mathbf{x}\}$

while $\mathbf{x}$ is not MNP(P)

$\quad \mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$

$\quad C = C \cup \{\mathbf{p}_j\}$

$\quad \mathbf{y} = \mathbf{MNP(aff(C))}$

$\quad$ while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$

$$\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$$

$\quad\quad C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of $\text{conv}(C)$

$\quad\quad \mathbf{x} = \mathbf{z}$

$\quad\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

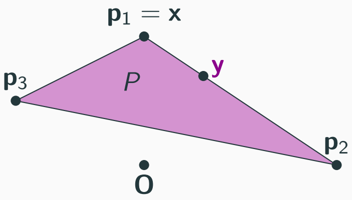
$\quad \mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$

$\mathbf{p}_2 = (3, 0)$

$\mathbf{p}_3 = (-2, 1)$

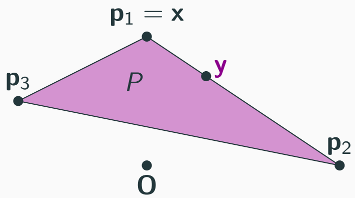


$C = \{\mathbf{p}_1, \mathbf{p}_2\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
  $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
  $C = C \cup \{\mathbf{p}_j\}$
  $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  while $\mathbf{y} \notin$ relint(conv(C))
    $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
    $\mathbf{x} = \mathbf{z}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_1, \mathbf{p}_2\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$

$C = \{\mathbf{x}\}$

while $\mathbf{x}$ is not MNP(P)

$\quad \mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$

$\quad C = C \cup \{\mathbf{p}_j\}$

$\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

$\quad$ while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$

$\quad\quad \mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$

$\quad\quad C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)

$\quad\quad \mathbf{x} = \mathbf{z}$

$\quad\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

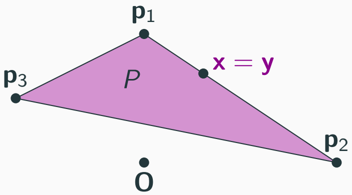
$\quad \mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$

$\mathbf{p}_2 = (3, 0)$

$\mathbf{p}_3 = (-2, 1)$

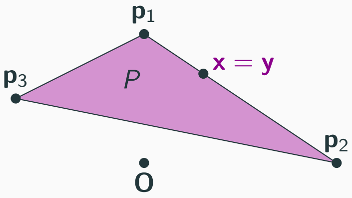


$C = \{\mathbf{p}_1, \mathbf{p}_2\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while x is not MNP(P)
  $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
  $C = C \cup \{\mathbf{p}_j\}$
  $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
    $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of $\text{conv}(C)$
    $\mathbf{x} = \mathbf{z}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_1, \mathbf{p}_2\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$

$C = \{\mathbf{x}\}$

while $\mathbf{x}$ is not MNP(P)

$\quad \mathbf{p_j} \in \{\mathbf{p} \in \mathbf{P} : \mathbf{x^T p} < ||\mathbf{x}||_2^2\}$

$\quad C = C \cup \{\mathbf{p}_j\}$

$\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

$\quad$ while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$

$$\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$$

$\qquad C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)

$\qquad \mathbf{x} = \mathbf{z}$

$\qquad \mathbf{y} = \text{MNP}(\text{aff}(C))$

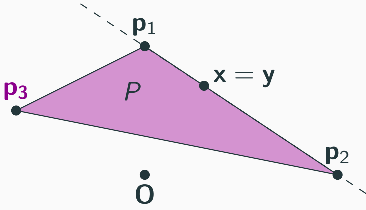
$\quad \mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$

$\mathbf{p}_2 = (3, 0)$

$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_1, \mathbf{p}_2\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$

$C = \{\mathbf{x}\}$

while $\mathbf{x}$ is not MNP(P)

  $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$

  $\mathbf{C} = \mathbf{C} \cup \{\mathbf{p_j}\}$

  $\mathbf{y} = \text{MNP}(\text{aff}(C))$

  while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$

    $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$

    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of $\text{conv}(C)$

    $\mathbf{x} = \mathbf{z}$

    $\mathbf{y} = \text{MNP}(\text{aff}(C))$

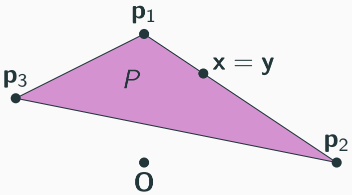
  $\mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$

$\mathbf{p}_2 = (3, 0)$

$\mathbf{p}_3 = (-2, 1)$



$\mathbf{C} = \{\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)

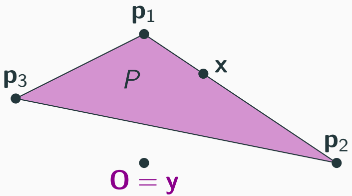
    $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
    $C = C \cup \{\mathbf{p}_j\}$
    $\mathbf{y} = \mathbf{MNP(aff(C))}$
    while $\mathbf{y} \notin$ relint(conv(C))

        $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
        $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
        $\mathbf{x} = \mathbf{z}$
        $\mathbf{y} = \text{MNP}(\text{aff}(C))$

    $\mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
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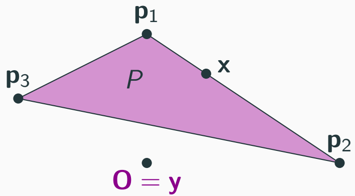
while $\mathbf{x}$ is not MNP(P)
  $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
  $C = C \cup \{\mathbf{p}_j\}$
  $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  while $\mathbf{y} \notin$ relint(conv(C))
    $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
    $\mathbf{x} = \mathbf{z}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$

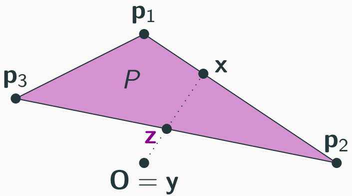


$C = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
    $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
    $C = C \cup \{\mathbf{p}_j\}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
        $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(\mathbf{C}) \cap \overline{\mathbf{xy}}}{\textbf{argmin}} ||\mathbf{z} - \mathbf{y}||_2$
        $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
        $\mathbf{x} = \mathbf{z}$
        $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$

$C = \{\mathbf{x}\}$

while $\mathbf{x}$ is not MNP(P)

$\quad \mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$

$\quad C = C \cup \{\mathbf{p}_j\}$

$\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

$\quad$ while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$

$\quad\quad \mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$

$\quad\quad$ **$\mathbf{C} = \mathbf{C} - \{\mathbf{p_i}\}$ where $\mathbf{p_i}, \mathbf{z}$ are on different faces of conv($\mathbf{C}$)**

$\quad\quad \mathbf{x} = \mathbf{z}$

$\quad\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

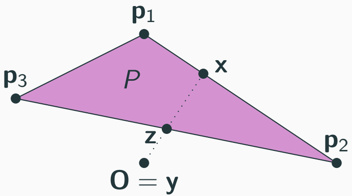
$\quad \mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$

$\mathbf{p}_2 = (3, 0)$

$\mathbf{p}_3 = (-2, 1)$



$\mathbf{C} = \{\mathbf{p_2}, \mathbf{p_3}\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$

$C = \{\mathbf{x}\}$

while $\mathbf{x}$ is not MNP(P)

$\quad \mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$

$\quad C = C \cup \{\mathbf{p}_j\}$

$\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

$\quad$ while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$

$\quad\quad \mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$

$\quad\quad C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)

$\quad\quad \mathbf{x} = \mathbf{z}$

$\quad\quad \mathbf{y} = \text{MNP}(\text{aff}(C))$

$\quad \mathbf{x} = \mathbf{y}$

return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$

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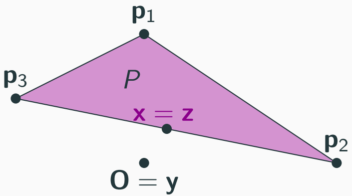
$\mathbf{p}_3 = (-2, 1)$

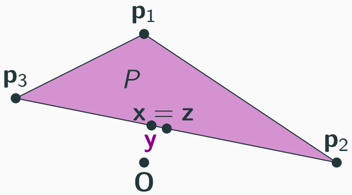


$C = \{\mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
 $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
 $C = C \cup \{\mathbf{p}_j\}$
 $\mathbf{y} = \text{MNP}(\text{aff}(C))$
 while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
  $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
  $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
  $\mathbf{x} = \mathbf{z}$
  $\mathbf{y} = \textbf{MNP}(\textbf{aff}(\textbf{C}))$
 $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
  $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
  $C = C \cup \{\mathbf{p}_j\}$
  $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  while $\mathbf{y} \notin$ relint(conv(C))
    $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
    $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of conv(C)
    $\mathbf{x} = \mathbf{z}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
  $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
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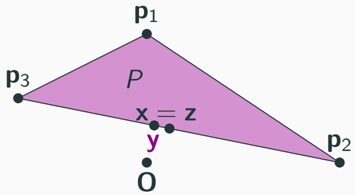
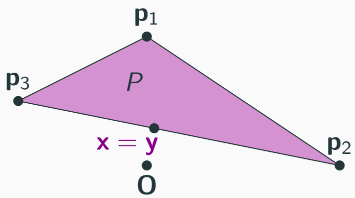


$C = \{\mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
    $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
    $C = C \cup \{\mathbf{p}_j\}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
        $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
        $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of $\text{conv}(C)$
        $\mathbf{x} = \mathbf{z}$
        $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$

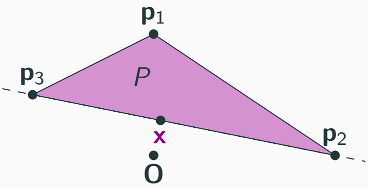


$C = \{\mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while x is not MNP(P)
    $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
    $C = C \cup \{\mathbf{p}_j\}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
        $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
        $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of $\text{conv}(C)$
        $\mathbf{x} = \mathbf{z}$
        $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    $\mathbf{x} = \mathbf{y}$
return $\mathbf{x}$

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$

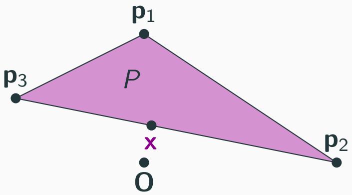


$C = \{\mathbf{p}_2, \mathbf{p}_3\}$

Sketch of Method

$\mathbf{x} \in P = \{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$
$C = \{\mathbf{x}\}$
while $\mathbf{x}$ is not MNP(P)
    $\mathbf{p}_j \in \{\mathbf{p} \in P : \mathbf{x}^T\mathbf{p} < \|\mathbf{x}\|_2^2\}$
    $C = C \cup \{\mathbf{p}_j\}$
    $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    while $\mathbf{y} \notin \text{relint}(\text{conv}(C))$
        $\mathbf{z} = \underset{\mathbf{z} \in \text{conv}(C) \cap \overline{\mathbf{xy}}}{\text{argmin}} \|\mathbf{z} - \mathbf{y}\|_2$
        $C = C - \{\mathbf{p}_i\}$ where $\mathbf{p}_i$, $\mathbf{z}$ are on different faces of $\text{conv}(C)$
        $\mathbf{x} = \mathbf{z}$
        $\mathbf{y} = \text{MNP}(\text{aff}(C))$
    $\mathbf{x} = \mathbf{y}$
return x

$\mathbf{p}_1 = (0, 2)$
$\mathbf{p}_2 = (3, 0)$
$\mathbf{p}_3 = (-2, 1)$



$C = \{\mathbf{p}_2, \mathbf{p}_3\}$

Wolfe's Method

$\mathbf{x} = \mathbf{p}_i$ for some $i = 1, 2, ..., m$, $\lambda = \mathbf{e}_i$

$C = \{i\}$

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Initial: `minnorm`
Insertion: `linopt` (select $\mathbf{p}_j$ minimizing $\mathbf{x}^T \mathbf{p}_j$), `minnorm`

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Exponential Behavior

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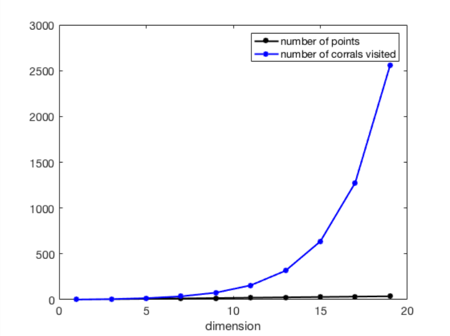

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$\xrightarrow[+4 \text{ points}]{+2 \text{ dim}}$

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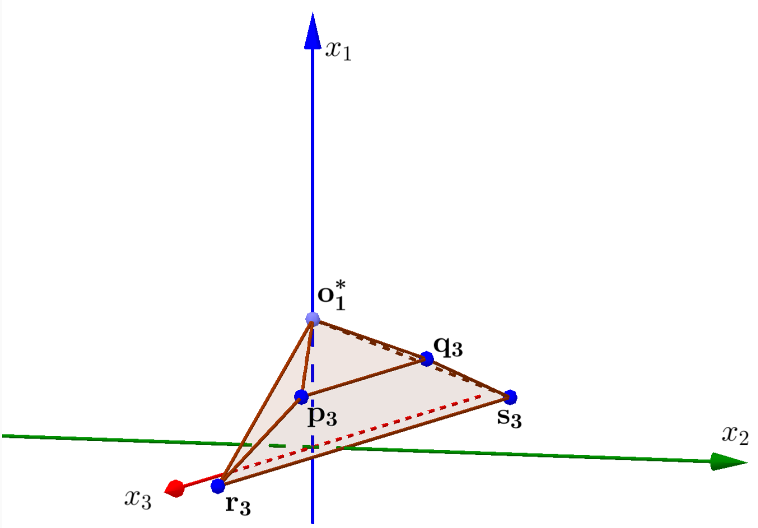
$P(1) := \{1\}$
$P(3) := \{(1,0,0), \mathbf{p}_3, \mathbf{q}_3, \mathbf{r}_3, \mathbf{s}_3\}$

Exponential Example: dim 3



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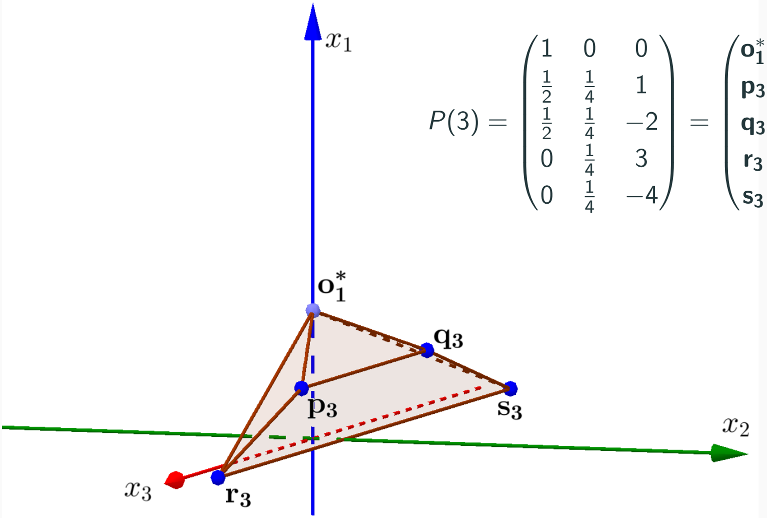

$$P(3) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 1 \\ \frac{1}{2} & \frac{1}{4} & -2 \\ 0 & \frac{1}{4} & 3 \\ 0 & \frac{1}{4} & -4 \end{pmatrix} = \begin{pmatrix} \mathbf{o}_1^* \\ \mathbf{p}_3 \\ \mathbf{q}_3 \\ \mathbf{r}_3 \\ \mathbf{s}_3 \end{pmatrix}$$

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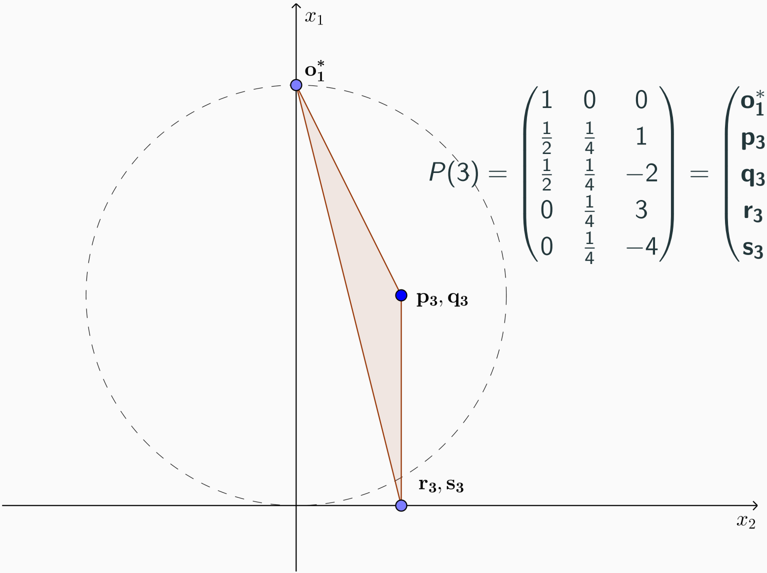

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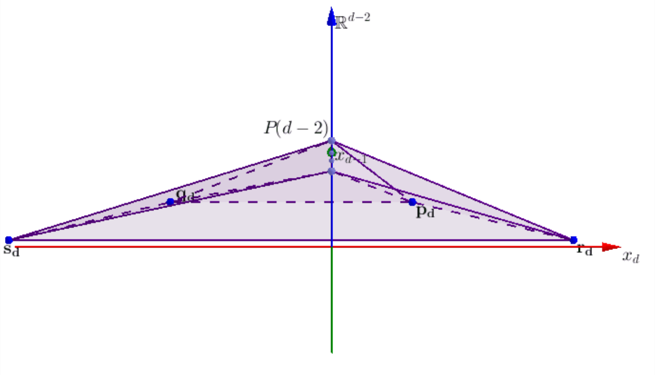

$$P(d) = \begin{pmatrix} P(d-2) & 0 & 0 \\ \frac{1}{2}\mathbf{o}^*_{\mathbf{d-2}} & \frac{m_{d-2}}{4} & M_{d-2} \\ \frac{1}{2}\mathbf{o}^*_{\mathbf{d-2}} & \frac{m_{d-2}}{4} & -(M_{d-2}+1) \\ 0 & \frac{m_{d-2}}{4} & M_{d-2}+2 \\ 0 & \frac{m_{d-2}}{4} & -(M_{d-2}+3) \end{pmatrix}$$

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$m_{d-2} = \|\mathbf{o}^*_{\mathbf{d-2}}\|_\infty$
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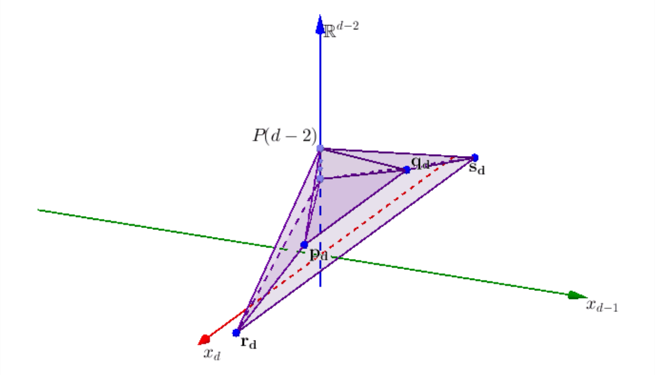

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Theorem (De Loera, H., Rademacher '17)

Consider the execution of Wolfe's method with the $\mathtt{minnorm}$ *insertion rule on input* $P(d)$ *where* $d = 2k - 1$. *Then the sequence of corrals,* $C(d)$ *has length* $5 \cdot 2^{k-1} - 4$.

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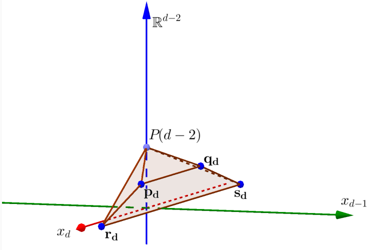
$C(d-2)$



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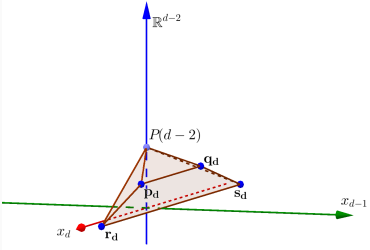

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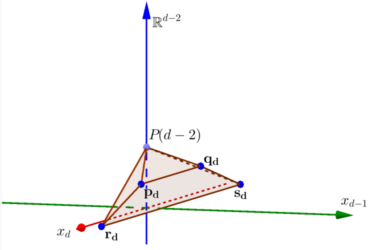
$$C(d-2) \longrightarrow \begin{array}{l} C(d-2) \\ O(d-2)\mathbf{p_d} \\ \mathbf{p_d q_d} \\ \mathbf{q_d r_d} \\ \mathbf{r_d s_d} \\ C(d-2)\mathbf{r_d s_d} \end{array}$$



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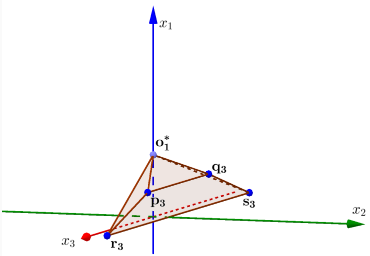

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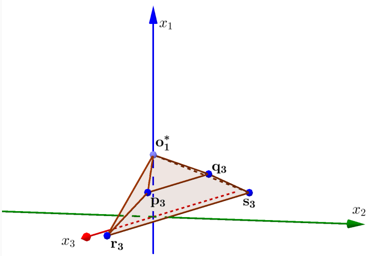

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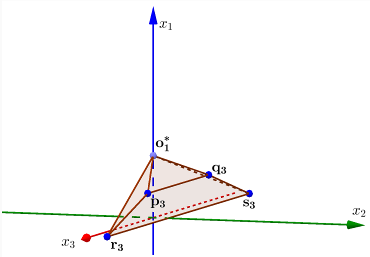

Adding Point to Corral

Lemma

Let $P \subseteq \mathbb{R}^d$ *be a finite set of points that is a corral. Let* $\mathbf{x}$ *be the minimum norm point in* $\operatorname{aff} P$*. Let* $\mathbf{q} \in \operatorname{span}\left(\mathbf{x}, \operatorname{span}(P)^{\perp}\right)$*, and assume* $\mathbf{q}^T\mathbf{x} < \min\{\|\mathbf{q}\|_2^2, \|\mathbf{x}\|_2^2\}$*. Then* $P \cup \{\mathbf{q}\}$ *is a corral. Moreover, the minimum norm point* $\mathbf{y}$ *in* $\operatorname{conv}(P \cup \{\mathbf{q}\})$ *is a (strict) convex combination of* $\mathbf{q}$ *and the minimum norm point of* P*:* $\mathbf{y} = \lambda\mathbf{x} + (1-\lambda)\mathbf{q}$ *with* $\lambda = \mathbf{q}^T(\mathbf{q}-\mathbf{x})/\|\mathbf{q}-\mathbf{x}\|_2^2$*.*

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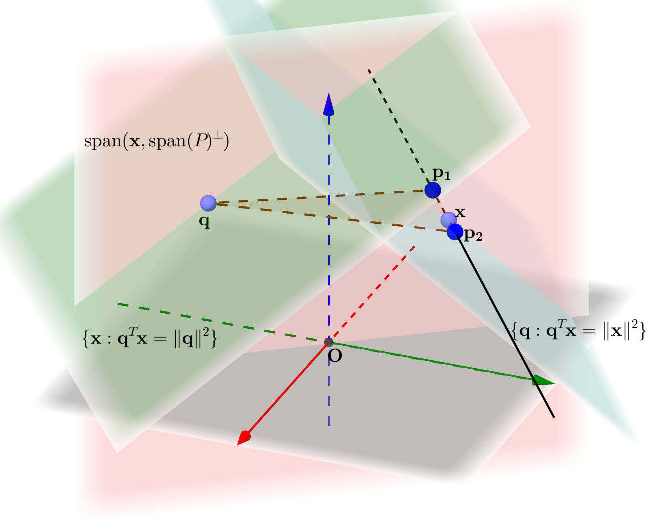

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Let $A \subseteq \mathbb{R}^d$ be a proper linear subspace. Let $P \subseteq A$ be a non-empty finite set. Let $Q \subseteq A^\perp$ be another non-empty finite set. Let $\mathbf{x}$ be the minimum norm point in $\operatorname{aff} P$*. Let $\mathbf{y}$ be the minimum norm point in* $\operatorname{aff} Q$*. Let $\mathbf{z}$ be the minimum norm point in* $\operatorname{aff}(P \cup Q)$*. We have:*

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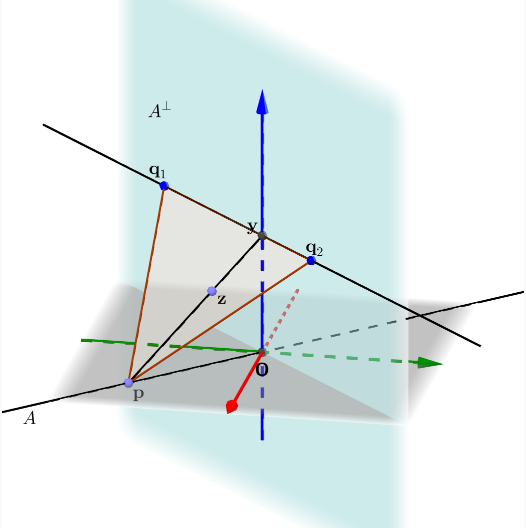
`the union of orthogonal corrals is still a corral`

Orthogonal Corrals



the union of orthogonal corrals is still a corral

Wolfe's Criterion under Addition of Orthogonal Point

Lemma

For a point $\mathbf{z}$ *define* $H_{\mathbf{z}} = \{\mathbf{w} \in \mathbb{R}^n : \mathbf{w} \cdot \mathbf{z} < \|\mathbf{z}\|_2^2\}$*. Suppose that we have an instance of the minimum norm point problem in* $\mathbb{R}^d$ *as follows: Some points,* P*, live in a proper linear subspace* A *and some,* Q*, in* $A^\perp$*. Let* $\mathbf{x}$ *be the minimum norm point in* $\operatorname{aff} P$ *and* $\mathbf{y}$ *be the minimum norm point in* $\operatorname{aff}(P \cup Q)$*. Then* $H_{\mathbf{y}} \cap A = H_{\mathbf{x}} \cap A$*.*

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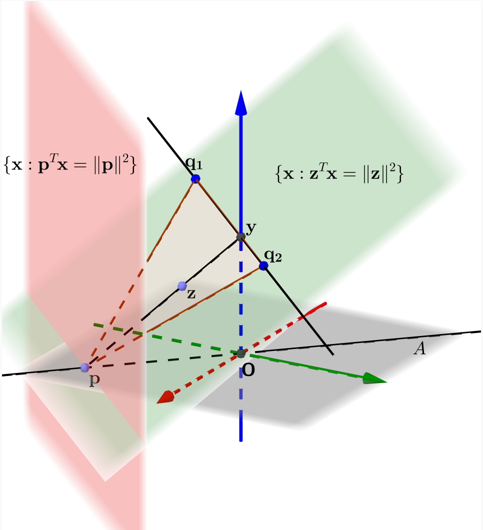
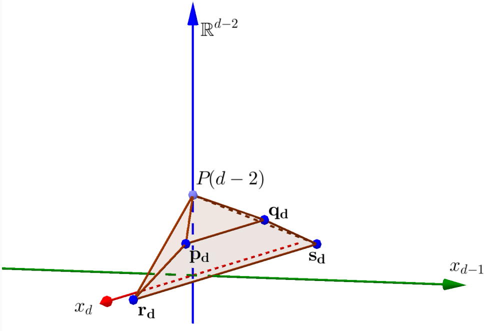


Sketch of Proof of Sequence $C(d)$: $C(d-2)$



$$P(d) = \begin{pmatrix} P(d-2) & 0 & 0 \\ \frac{1}{2}\mathbf{o}^*_{\mathbf{d-2}} & \frac{m_{d-2}}{4} & M_{d-2} \\ \frac{1}{2}\mathbf{o}^*_{\mathbf{d-2}} & \frac{m_{d-2}}{4} & -(M_{d-2}+1) \\ 0 & \frac{m_{d-2}}{4} & M_{d-2}+2 \\ 0 & \frac{m_{d-2}}{4} & -(M_{d-2}+3) \end{pmatrix}$$

$\mathbf{o}^*_{\mathbf{d-2}}$: MNP($P(d-2)$)

$m_{d-2} = \|\mathbf{o}^*_{\mathbf{d-2}}\|_\infty$

$M_{d-2} = \max_{\mathbf{p} \in P(d-2)} \|\mathbf{p}\|_1$

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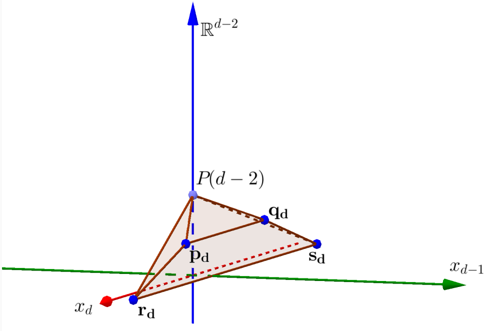

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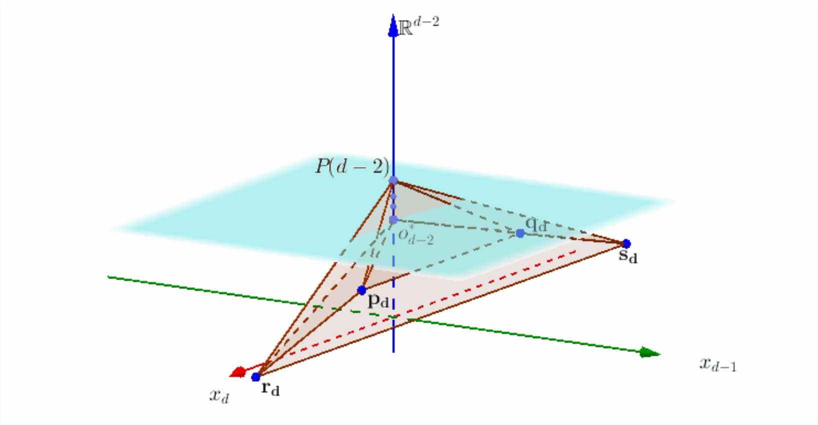
Sketch of Proof of Sequence $C(d)$: $O(d-2)\mathbf{p_d}$



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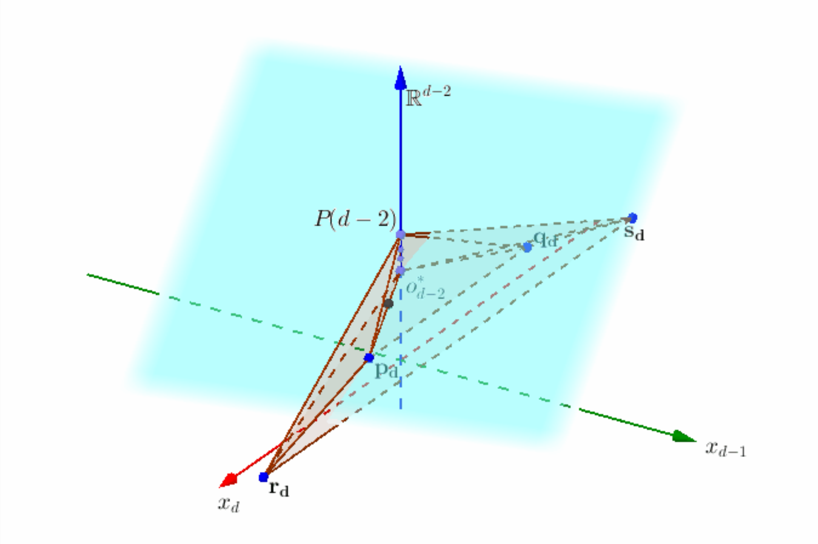
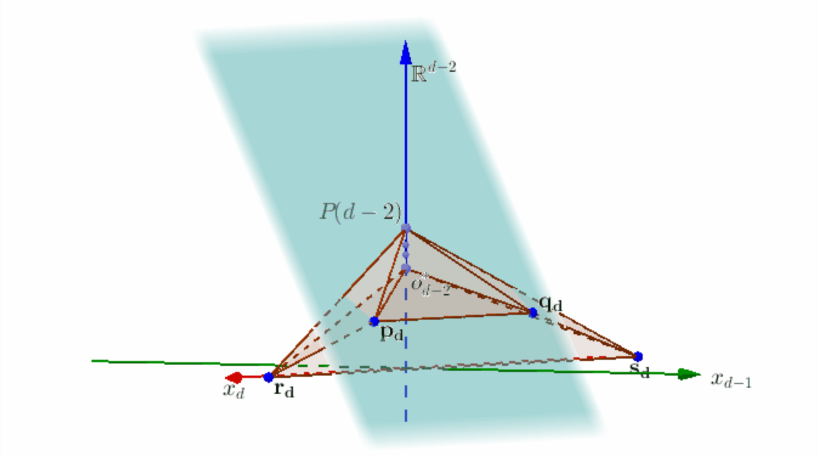


a corral with a point made from MNP and orthogonal directions is still a corral

Sketch of Proof of Sequence $C(d)$: $\mathbf{p_d q_d}$



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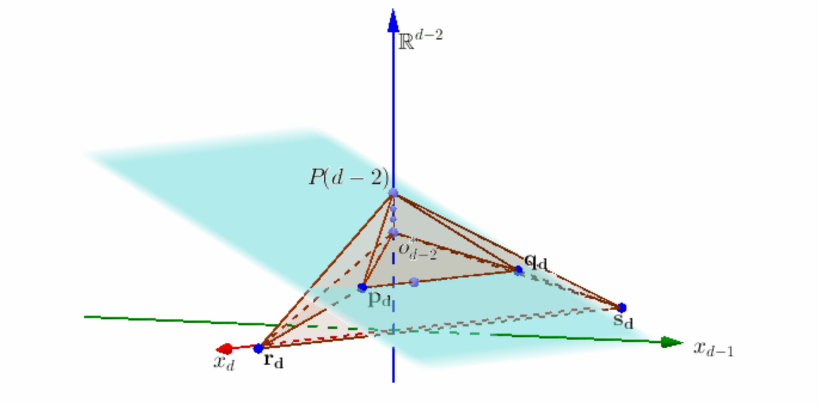
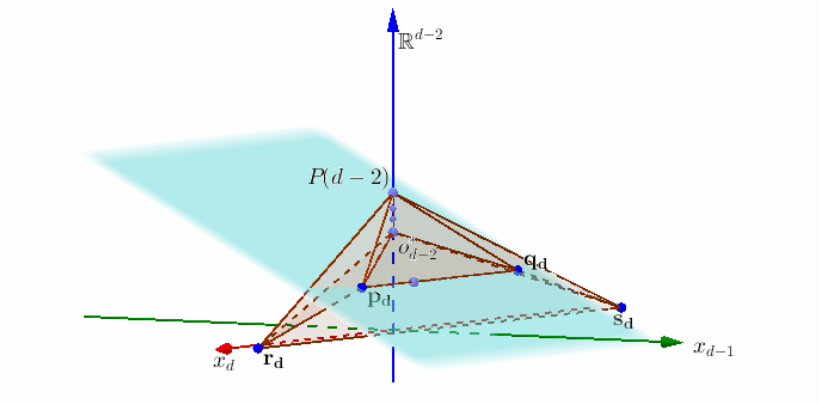


Sketch of Proof of Sequence $C(d)$: $\mathbf{q_d r_d}$



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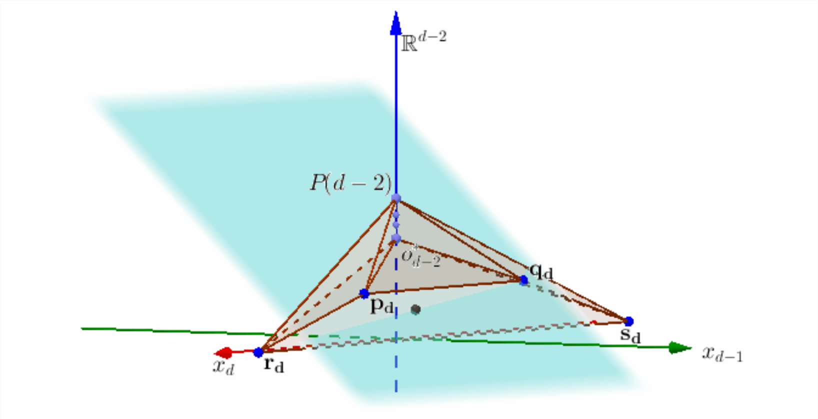
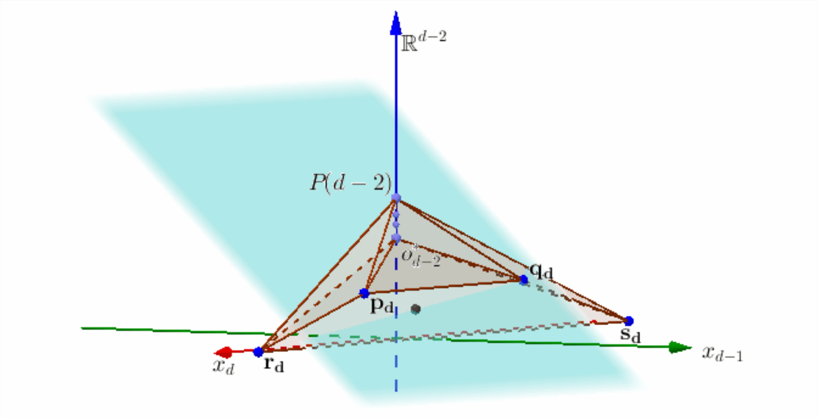


Sketch of Proof of Sequence $C(d)$: $\mathbf{r_d s_d}$



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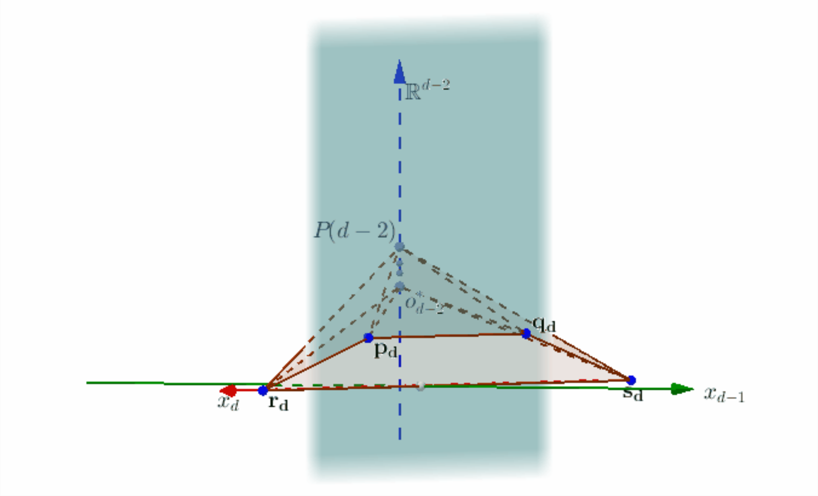
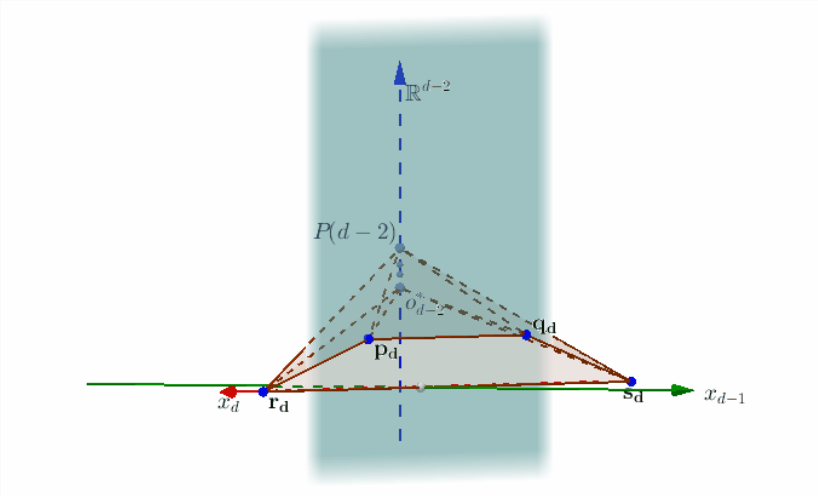


Sketch of Proof of Sequence $C(d)$: $C(d-2)\mathbf{r_d}\mathbf{s_d}$



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Conclusions

Future Directions

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3. Give an average (or smoothed) analysis of Wolfe's method.

Thanks...

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MATHEMATICS

Thanks for attending!

Questions?

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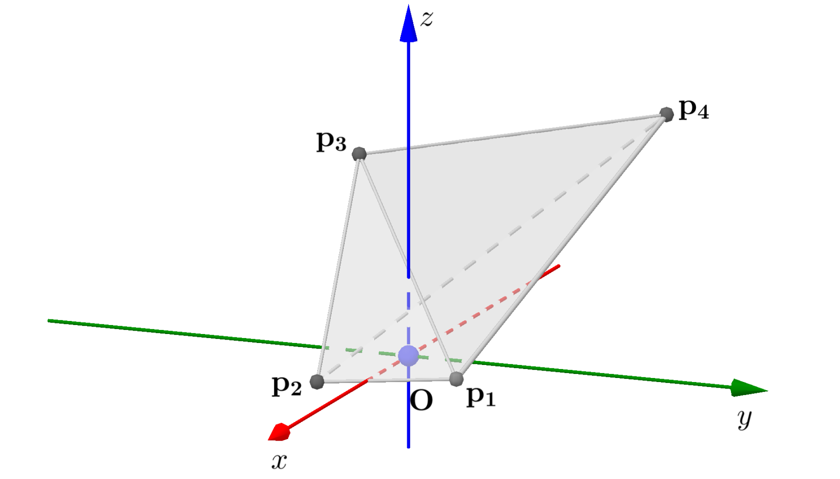
Citeseer, 2006.

Example: `minnorm` < `linopt`

$P = \text{conv}\{(0.8, 0.9, 0), (1.5, -0.5, 0), (-1, -1, 2), (-4, 1.5, 2)\} \subset \mathbb{R}^3$



Example: `minnorm` < `linopt`

Major Cycle	Minor Cycle	C
0	0	$\{p_1\}$
1	0	$\{p_1, p_2\}$
2	0	$\{p_1, p_2, p_3\}$
3	0	$\{p_1, p_2, p_3, p_4\}$
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