Connections between Iterative Methods for Linear Systems and Consensus Dynamics on Networks

CCMS Applied Mathematics Seminar

March 21st, 2022

Dr. Jamie Haddock

Department of Mathematics

Harvey Mudd College

Consensus dynamics on networks (e.g., average consensus).

Iterative methods for linear systems (e.g., Kaczmarz methods).

A **bridge** between consensus dynamics on networks and numerical linear algebra.

Benjamin Jarman
UCLA

Chen Yap
Planet Labs Inc.

JH, Benjamin Jarman, and Chen Yap (2022). Paving the Way for Consensus: Convergence of Block Gossip Algorithms. *Submitted*.

Hector Tierno
HMC

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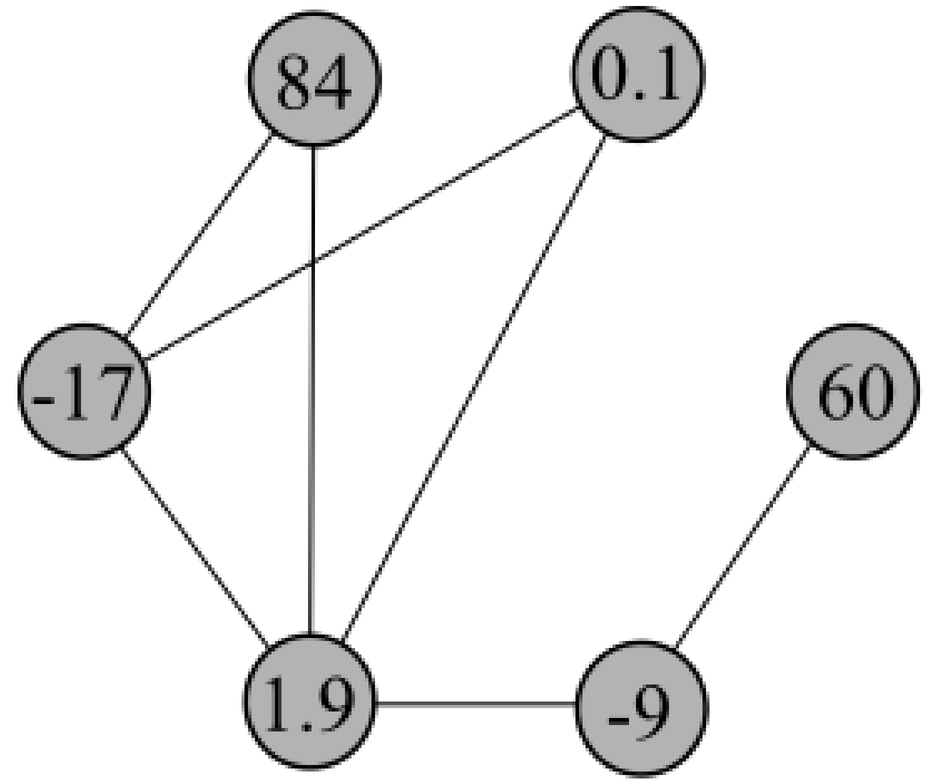

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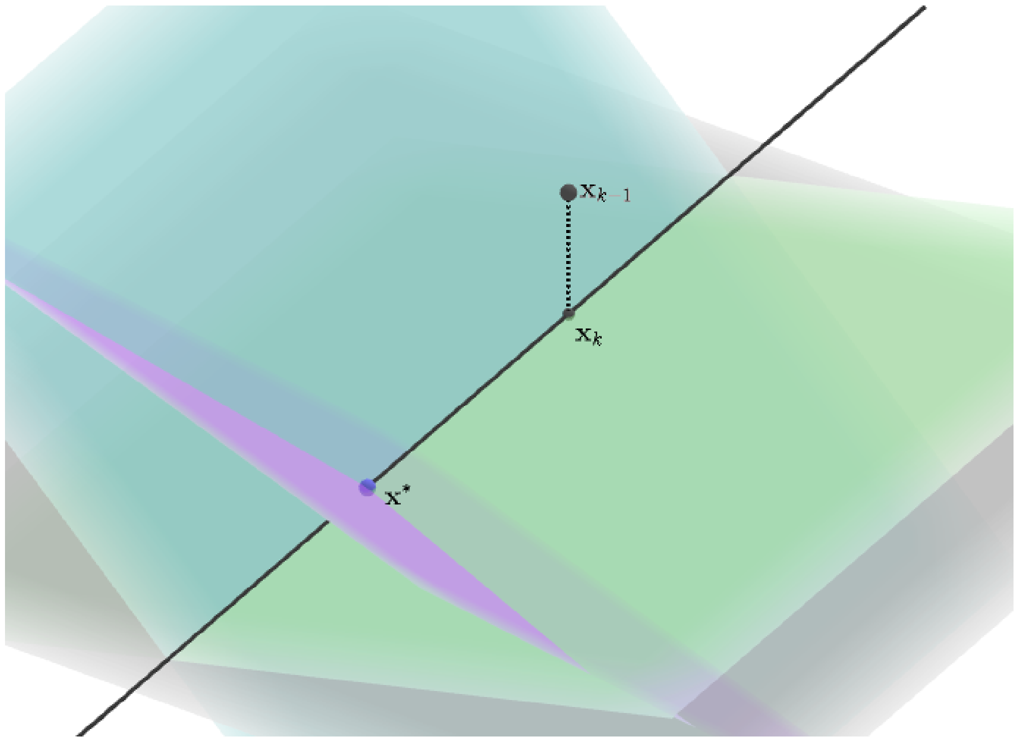

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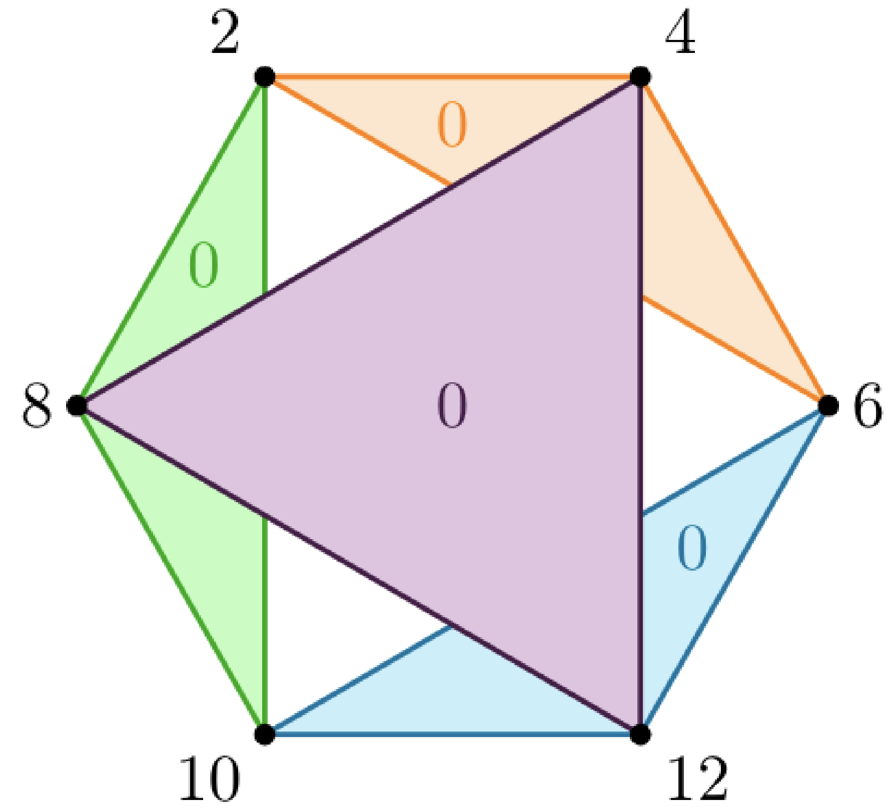

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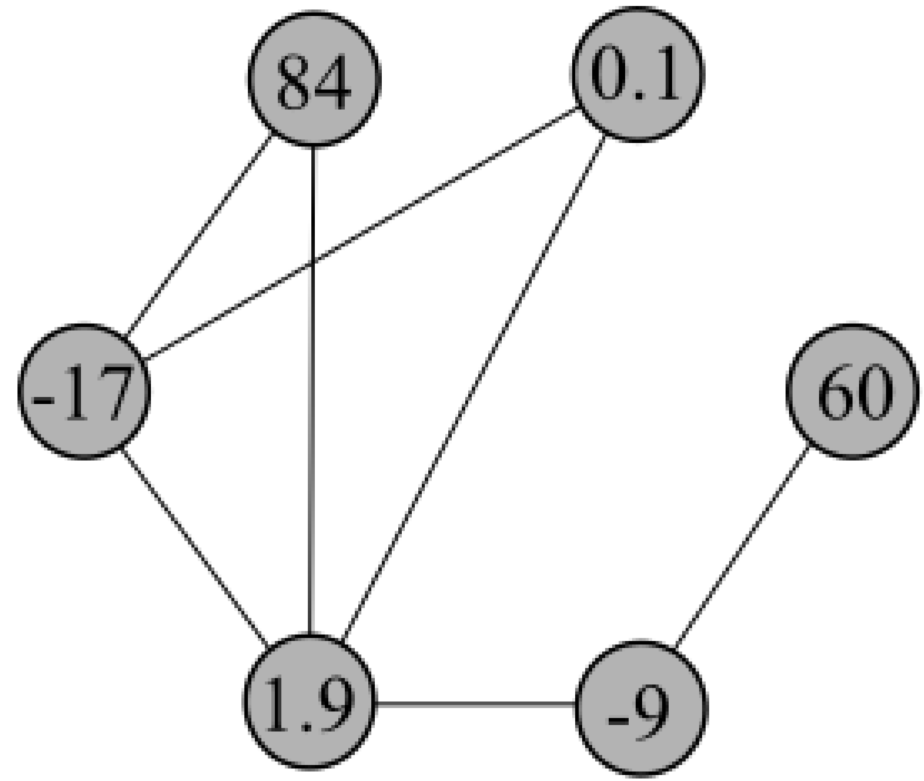
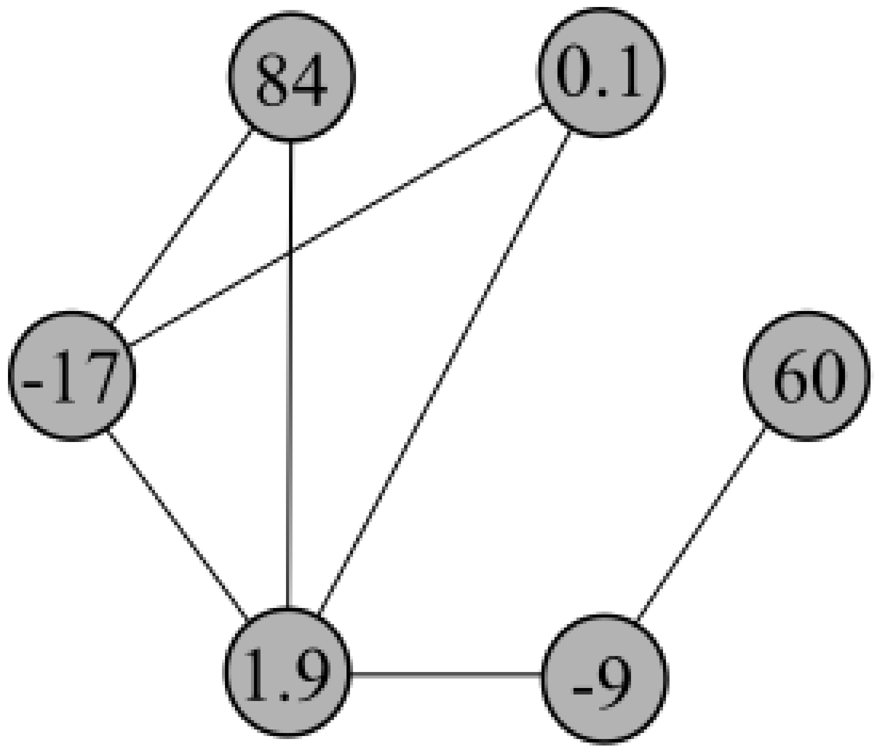


Consensus Dynamics

Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a graph with nodes $\mathcal{N}$ and edges $\mathcal{E}$.

Let $c_k(i)$ be a real scalar assigned to node i at time k.

Consensus dynamical systems are ones in which nodes values $c_k(i)$ evolve over time, i.e., they change their internal states according to some local interaction-rule, which is applied in every time step.



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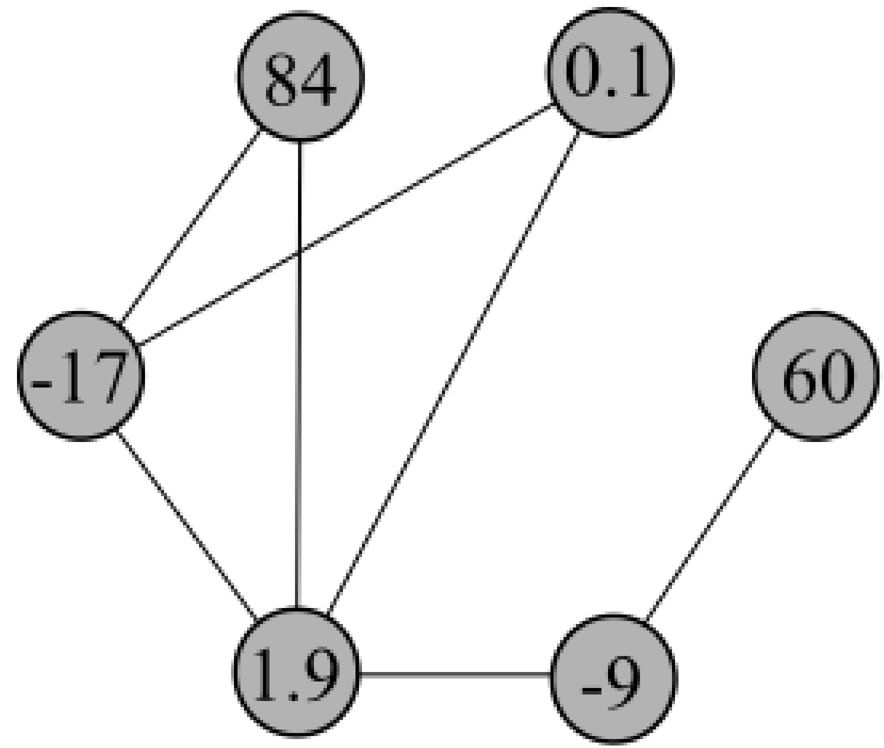

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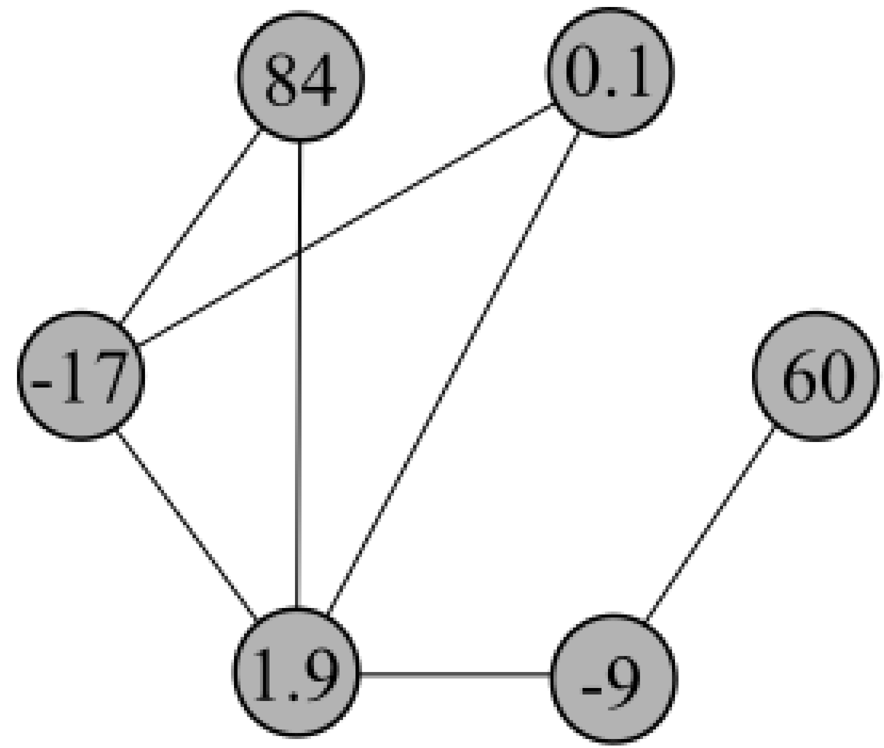
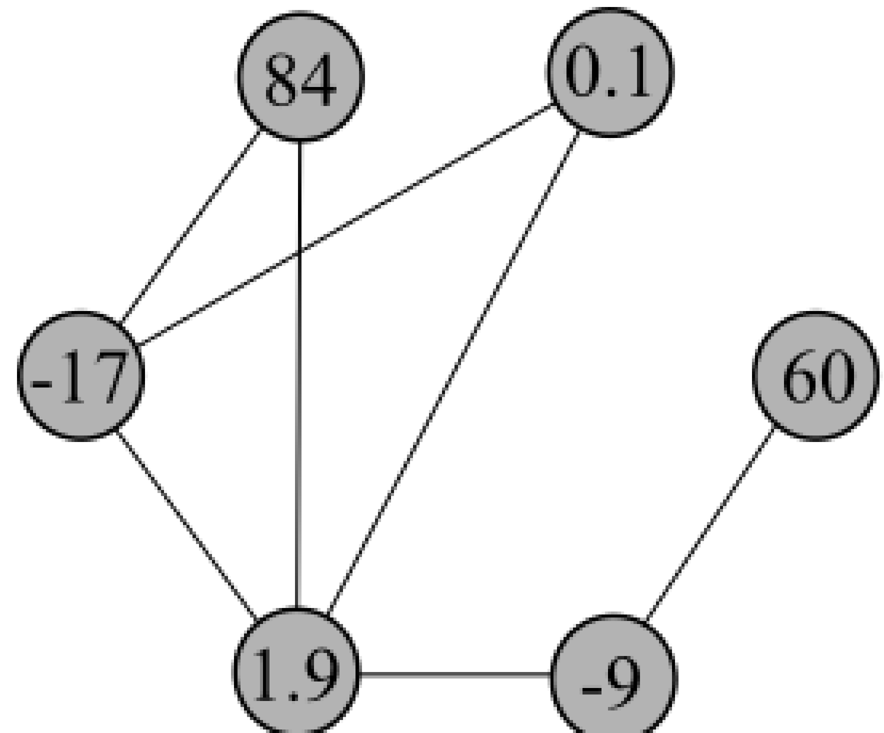


Consensus Dynamics: Applications

- opinion dynamics



Consensus Dynamics: Applications

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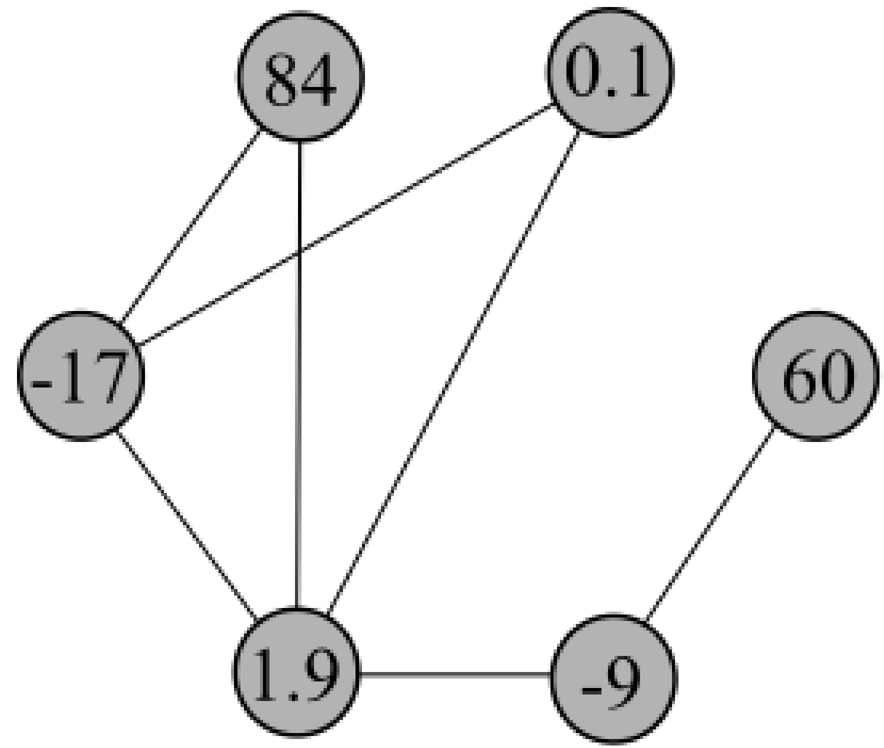
- voting and ranking models



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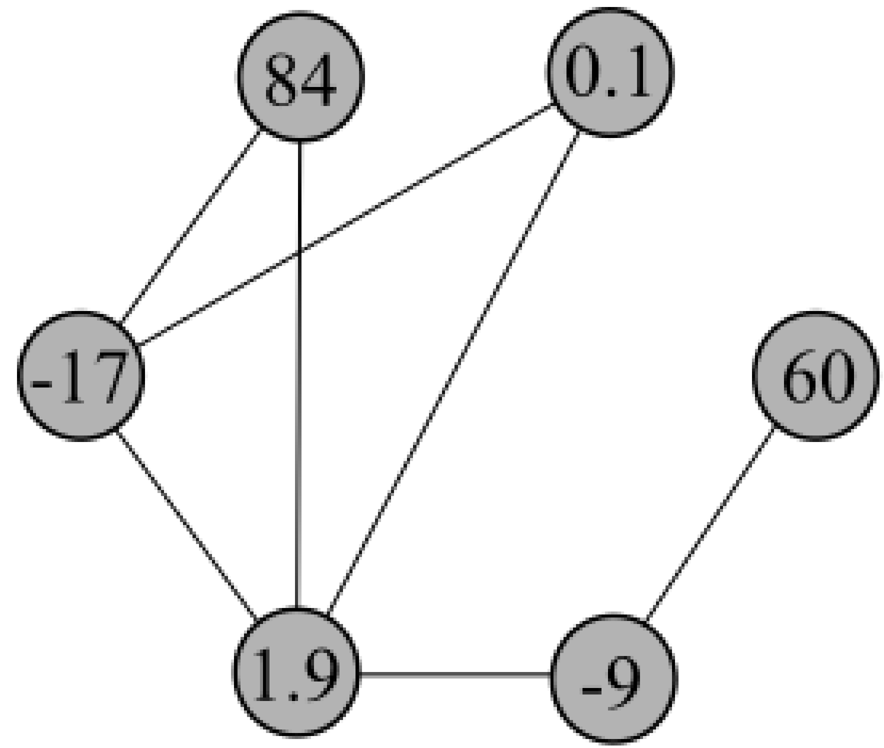

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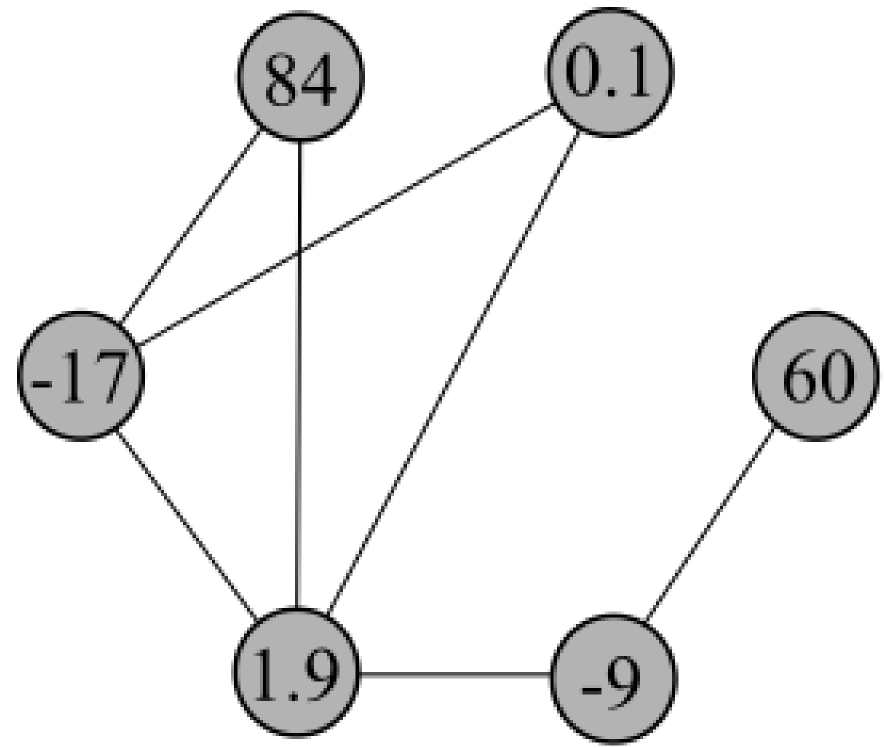

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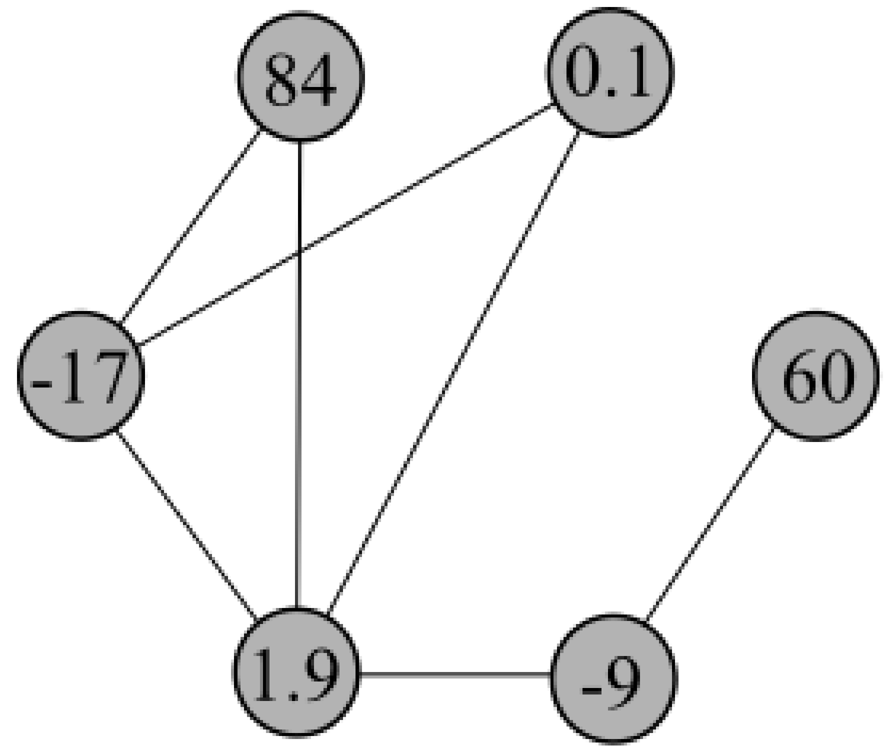

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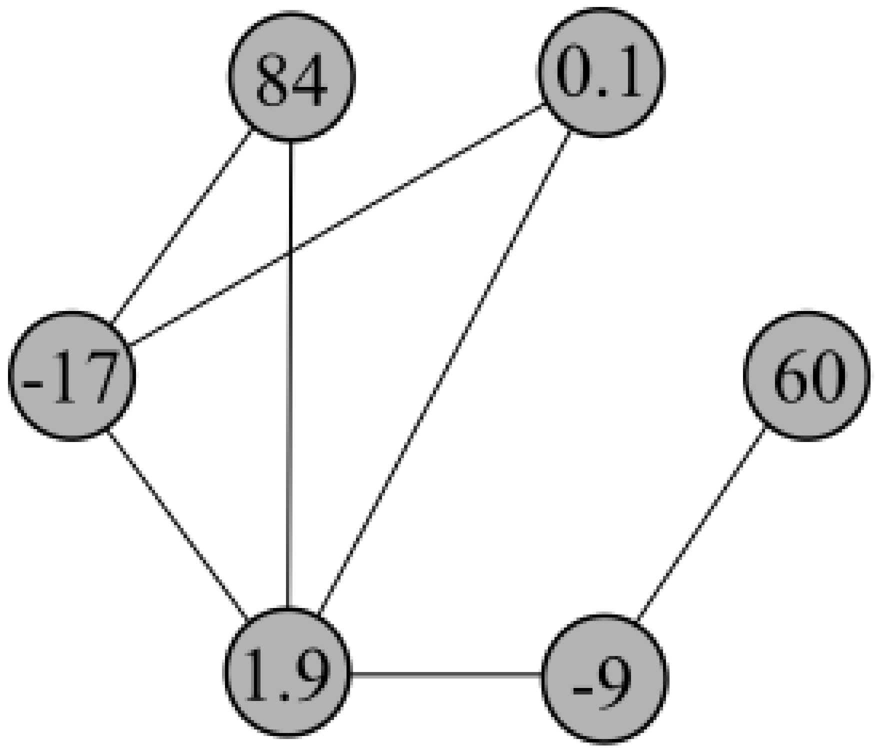

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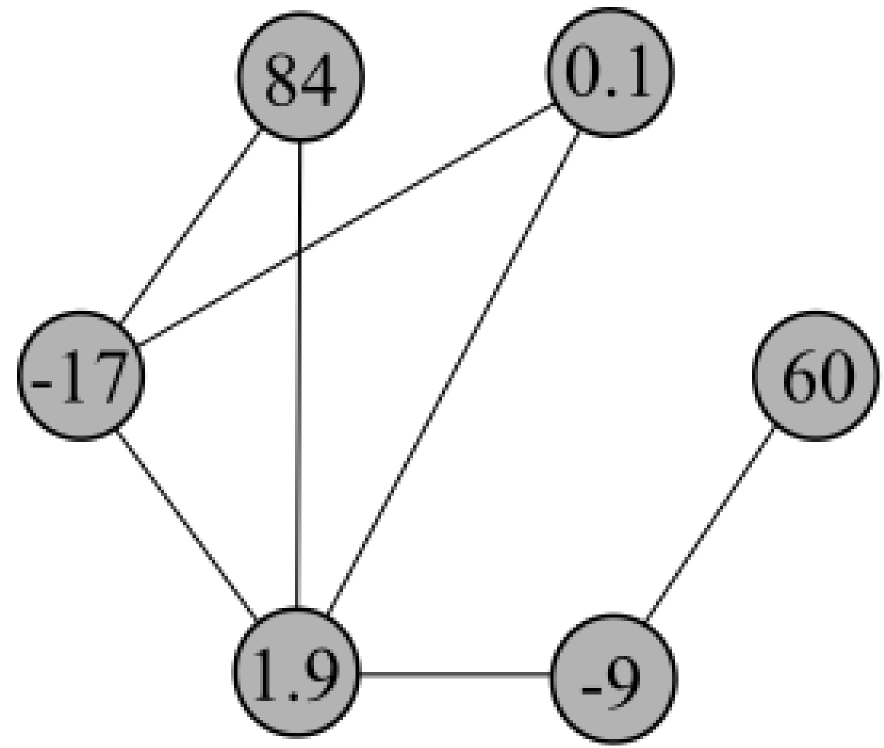

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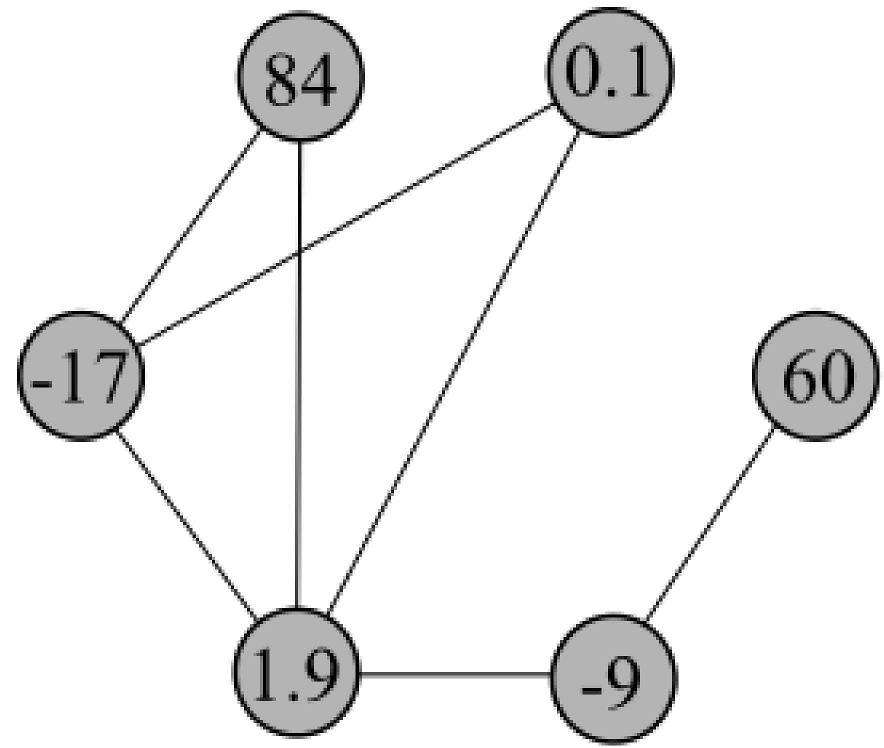
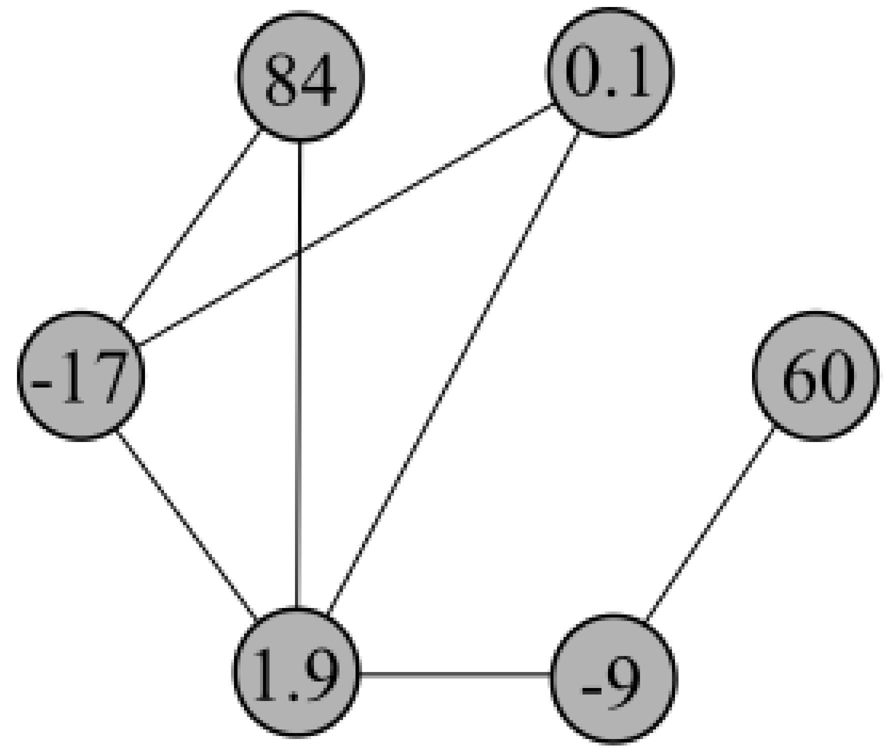


Consensus Dynamics: Model Types

- discrete state majority models



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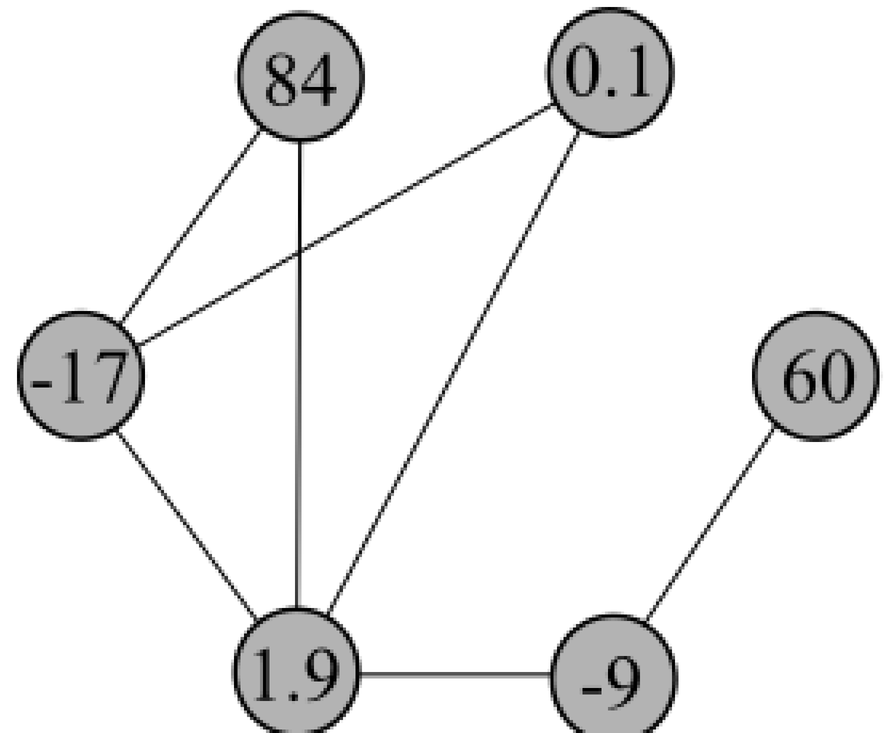

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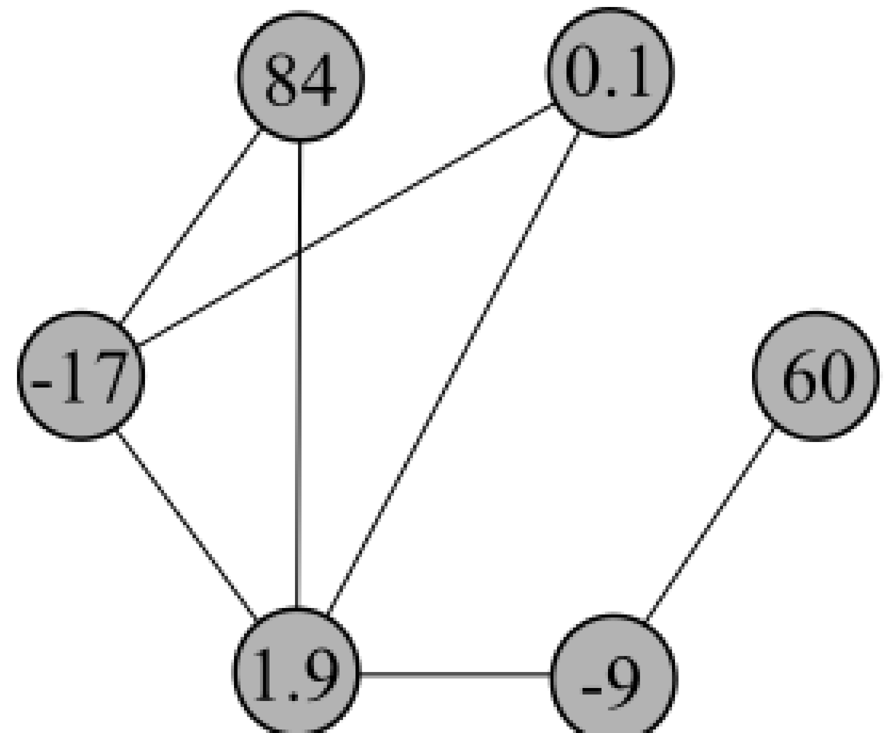

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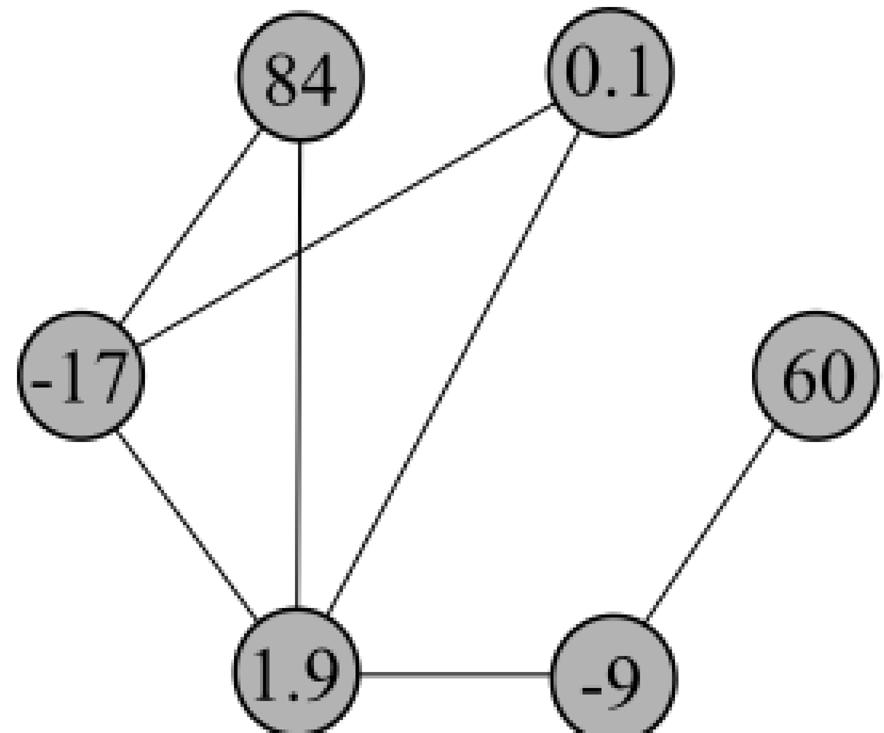
- **averaging models**



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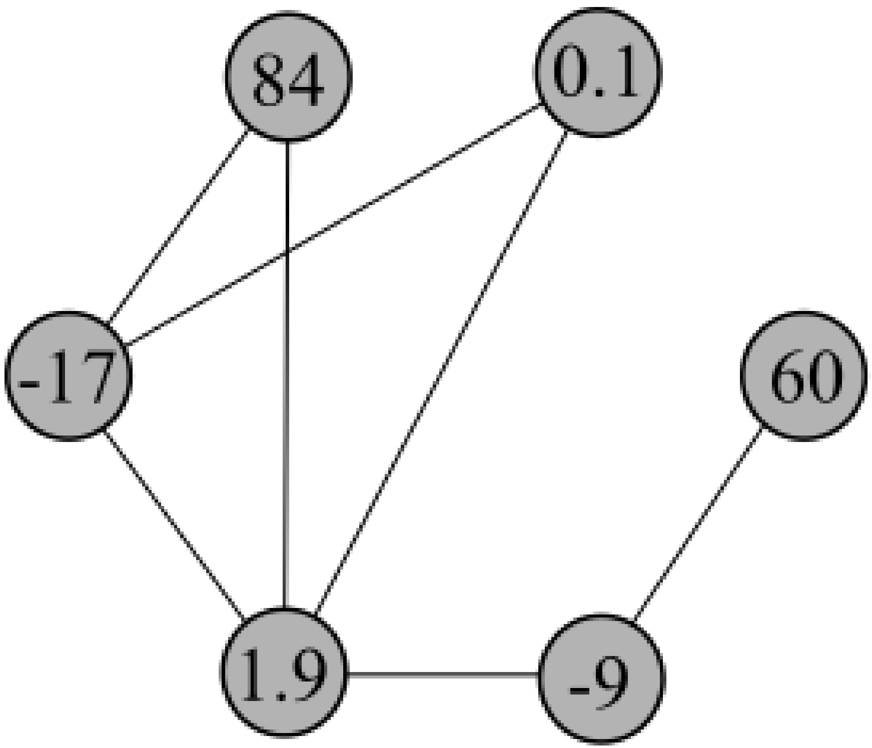
Example: Average Consensus

Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be an undirected connected graph with nodes $\mathcal{N}$ and edges $\mathcal{E}$.

Let $c_k(i)$ be a real scalar assigned to node i at time k.

The average consensus problem is to compute (iteratively) the average value $c^* := \sum_{i \in \mathcal{N}} c_0(i)/|\mathcal{N}|$ at every node, allowing only local communication on the graph.



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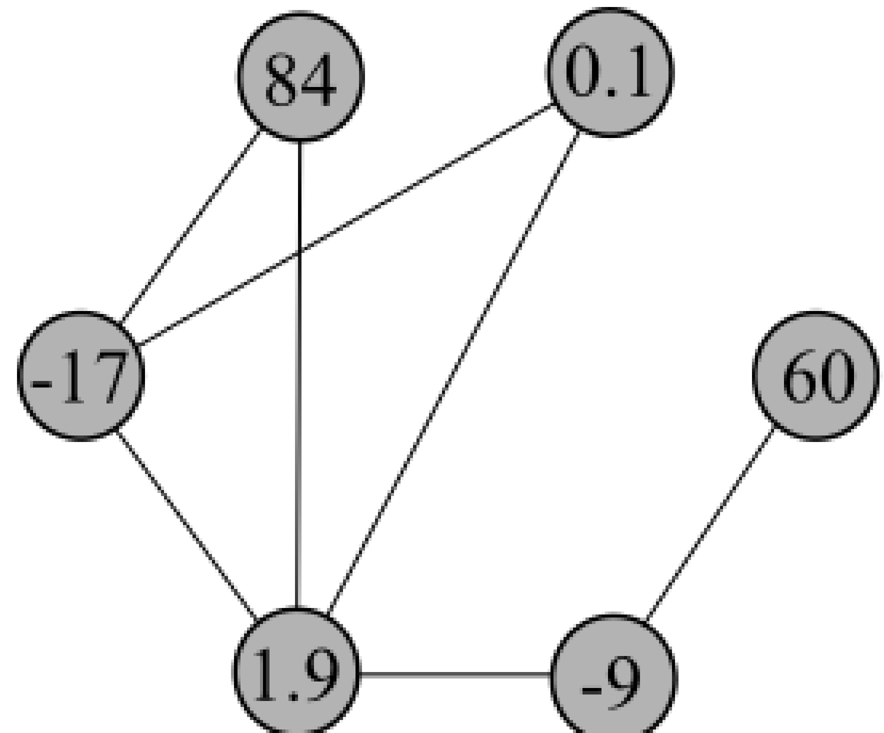

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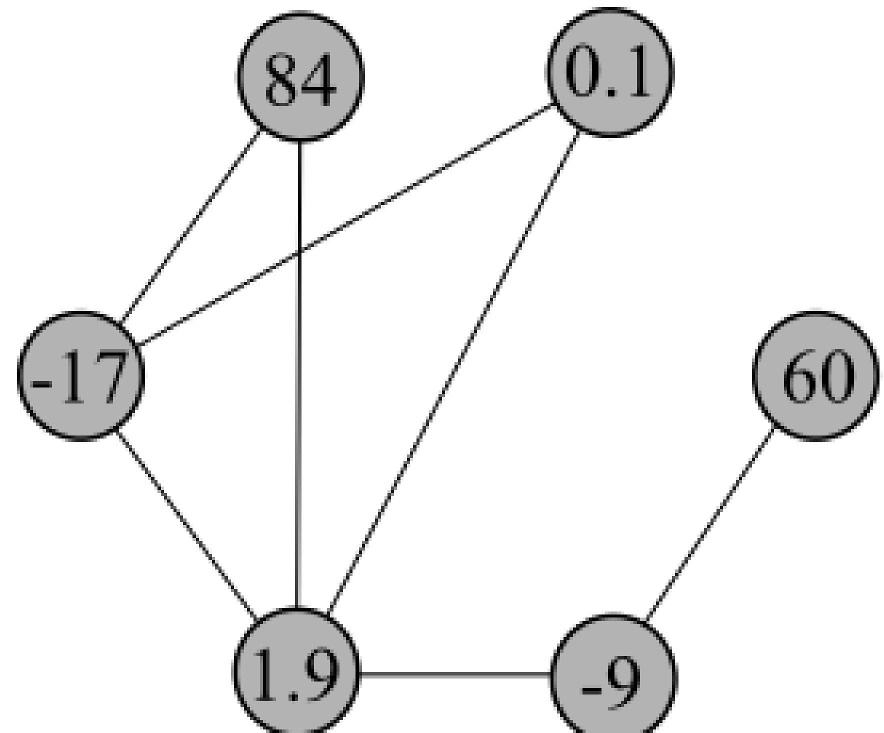

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- load balancing in parallel computing

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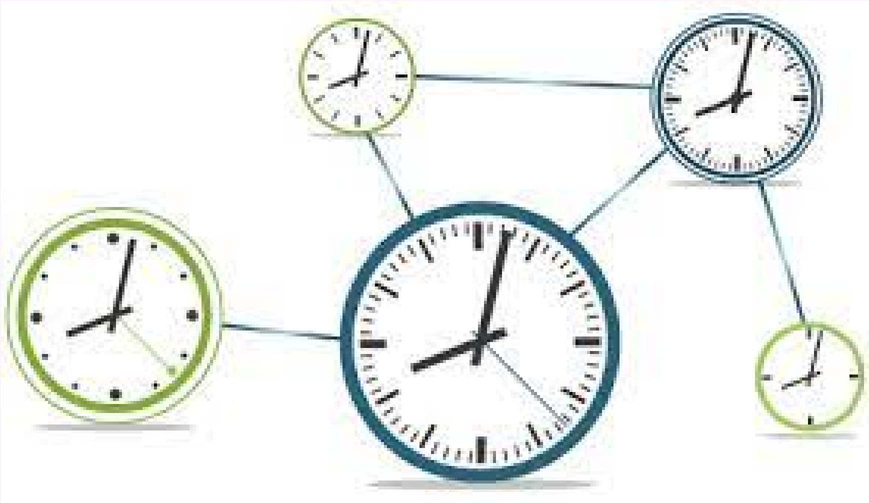

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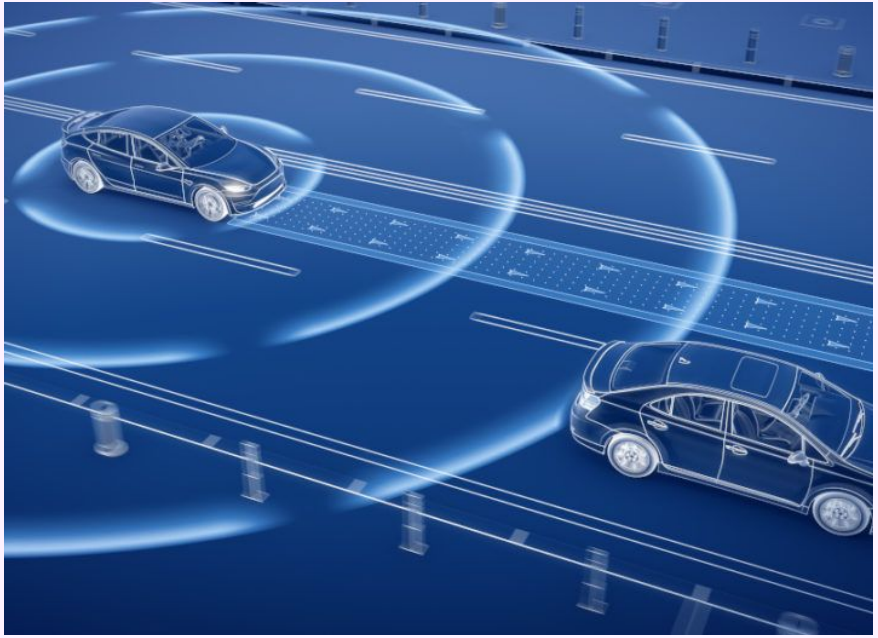
- coordination of mobile autonomous agents



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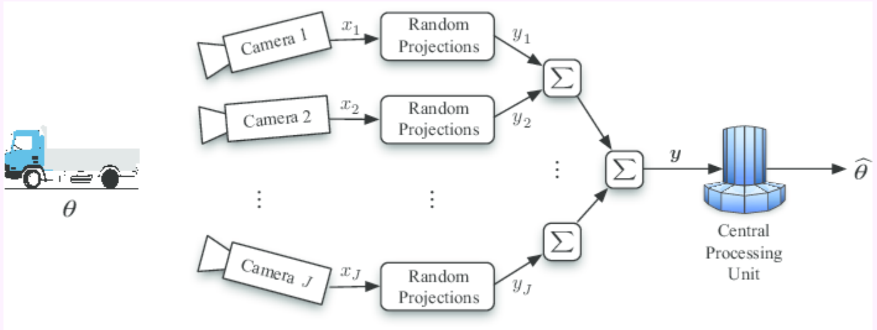

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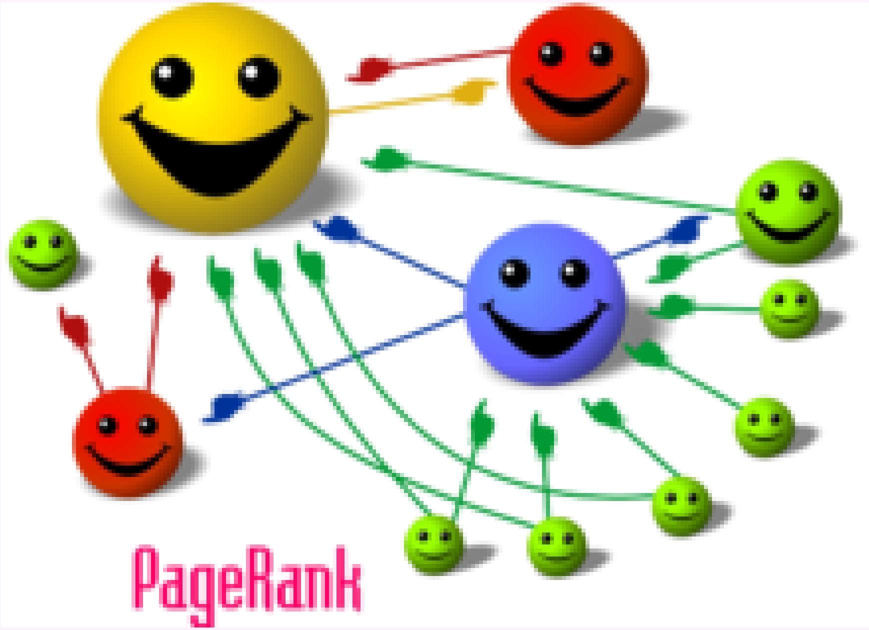

Average Consensus Applications

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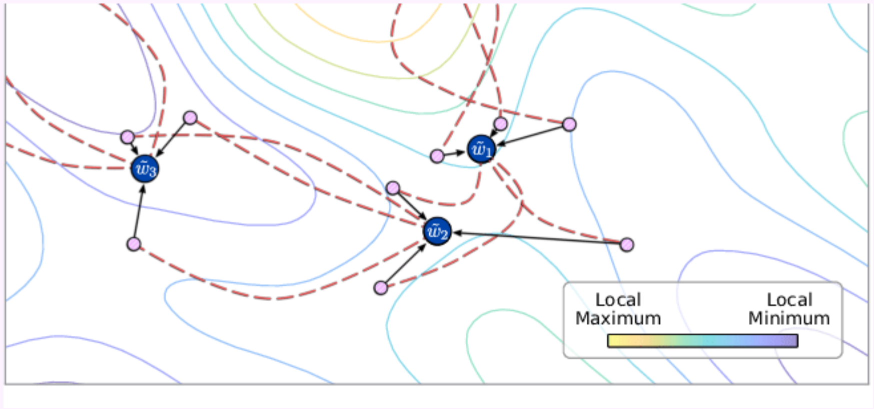

Block Gossip Method

Given graph $\mathcal{G}$, initial values $\mathbf{c}_0$, and edge subsets $T = \{\tau_1, \cdots, \tau_d\}$, for $k = 1, 2, \cdots$:

- Choose edge subset τ uniformly at random from T.
- Form $\mathcal{G}_\tau$, the edge-induced subgraph of $\mathcal{G}$ defined by edges in τ.
- Nodes in each connected component of $\mathcal{G}_\tau$ average their values and nodes outside of $\mathcal{G}_\tau$ do not update; this produces new secret values $\mathbf{c}_k$.

Related to the unbounded Deffuant–Weisbuch model.
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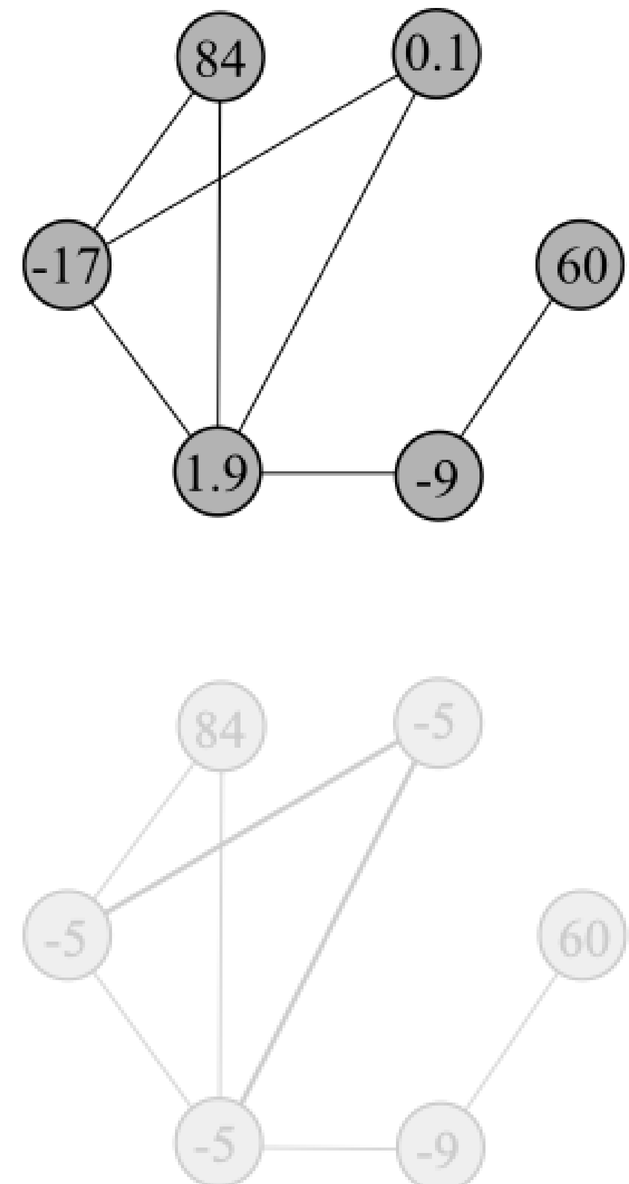

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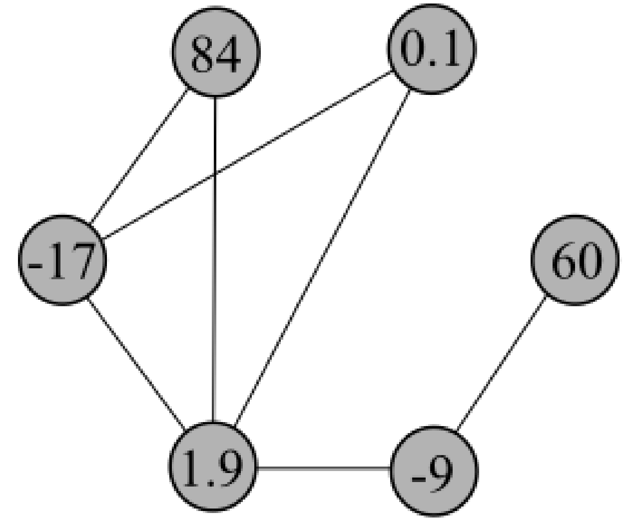

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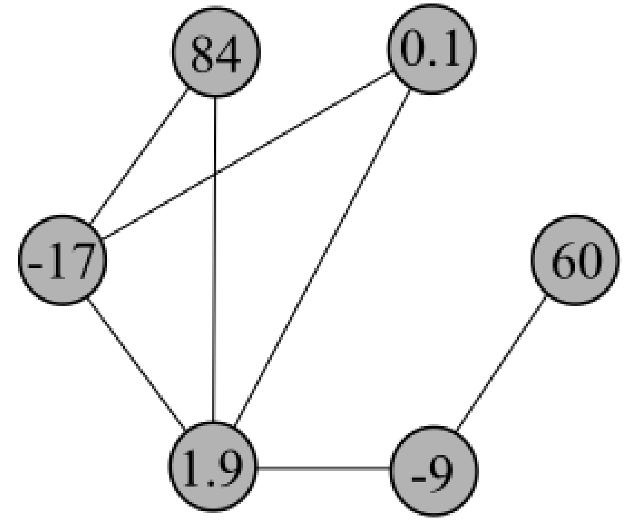

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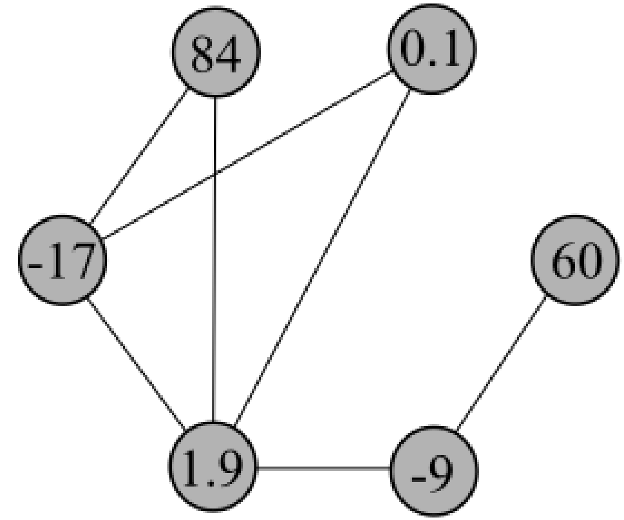

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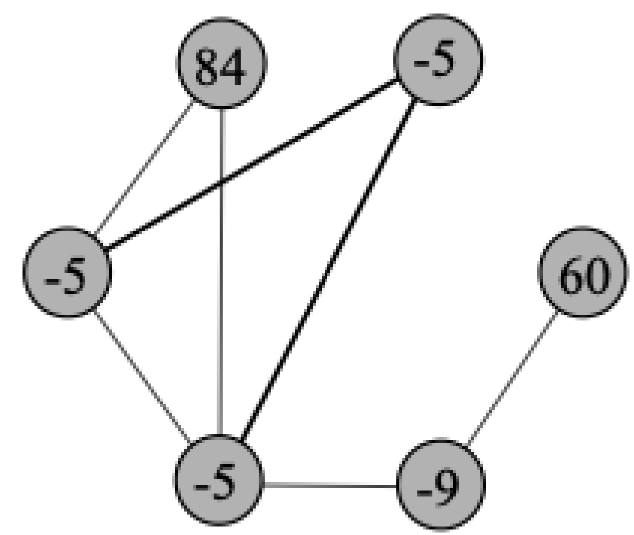

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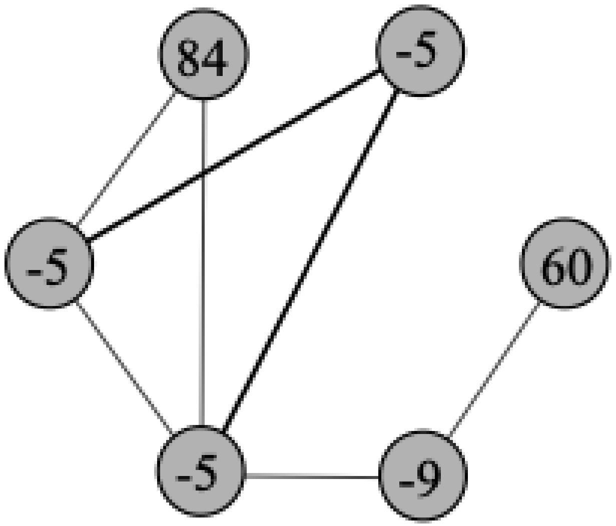

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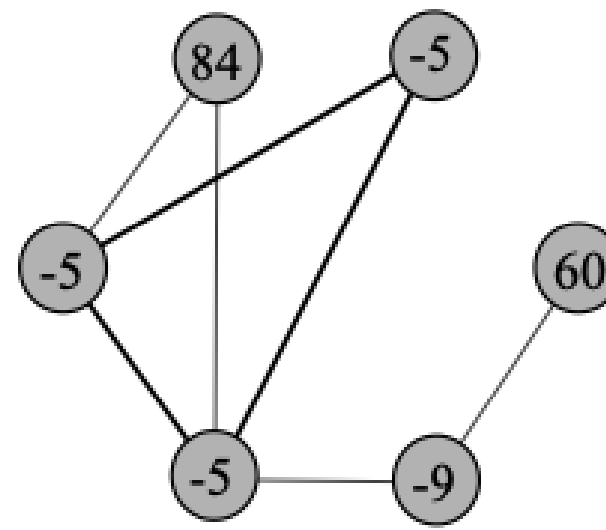

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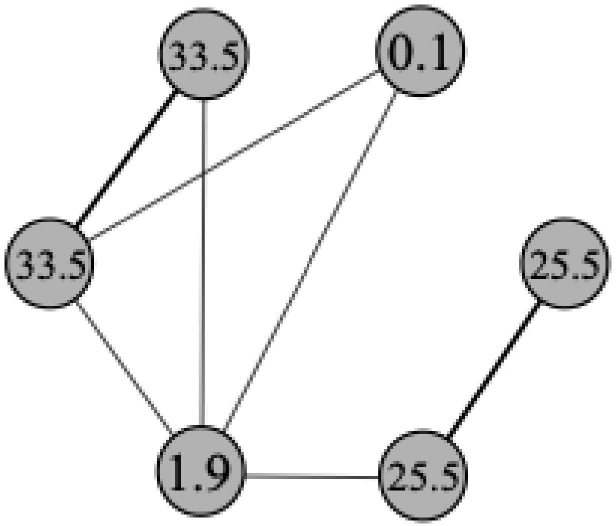

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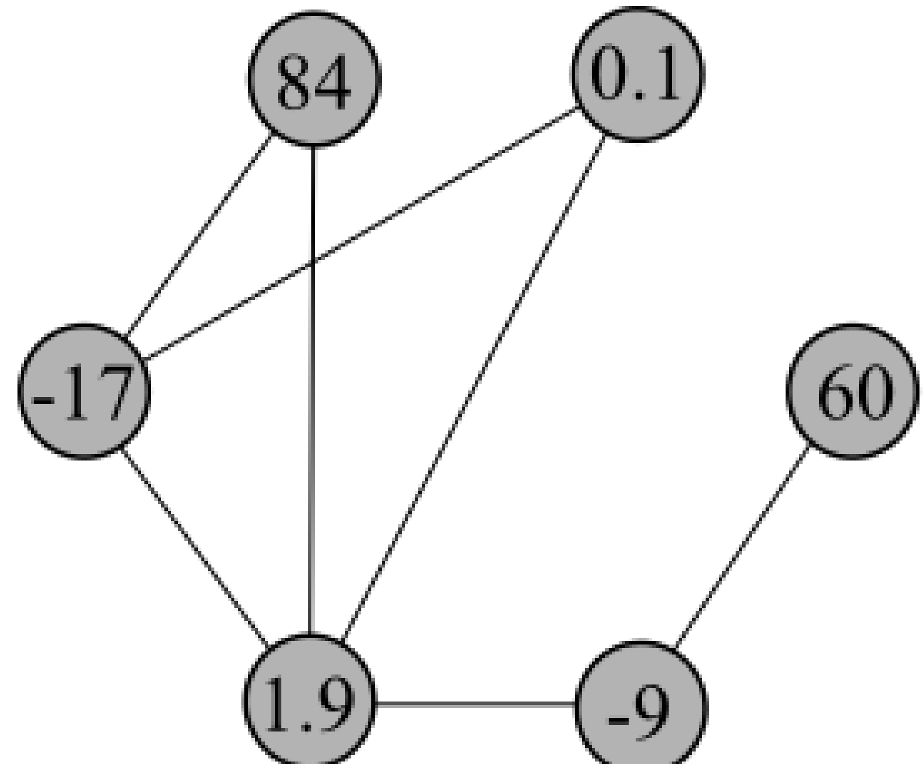

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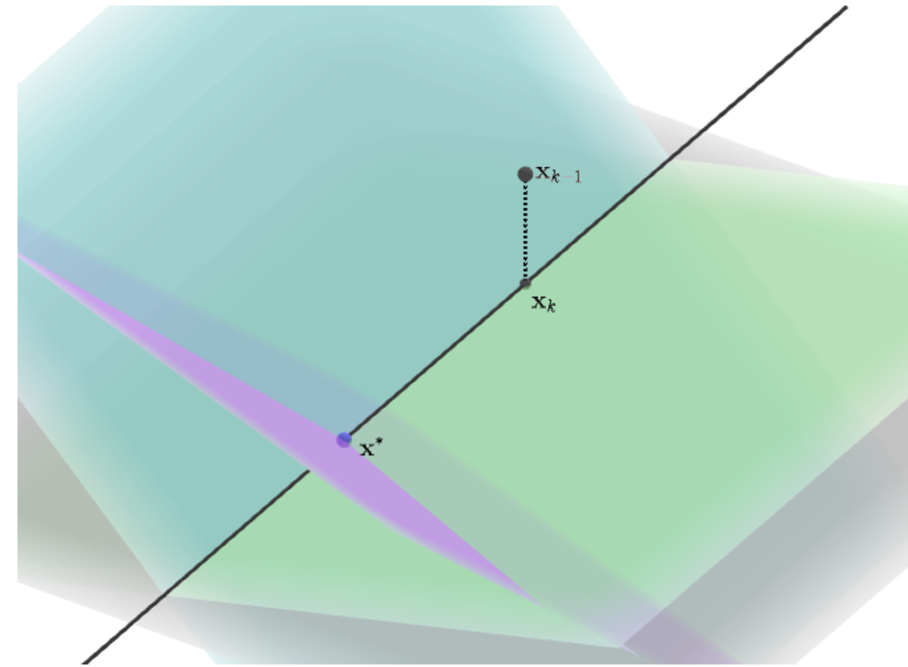
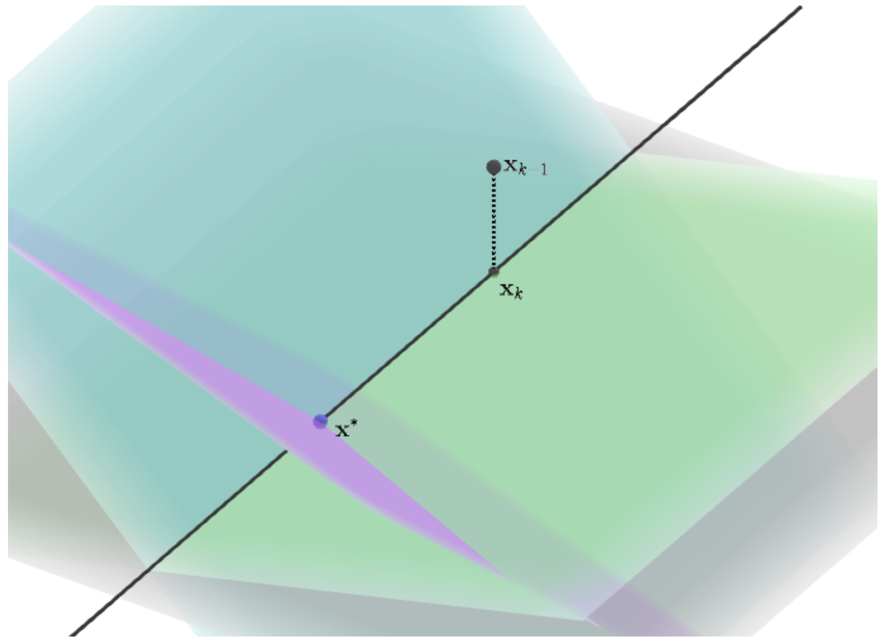


Iterative Methods for Linear Systems

Many classical numerical linear algebraic iterative methods for solving linear systems operate with row or column subset information, and/or entry-wise on iterates.



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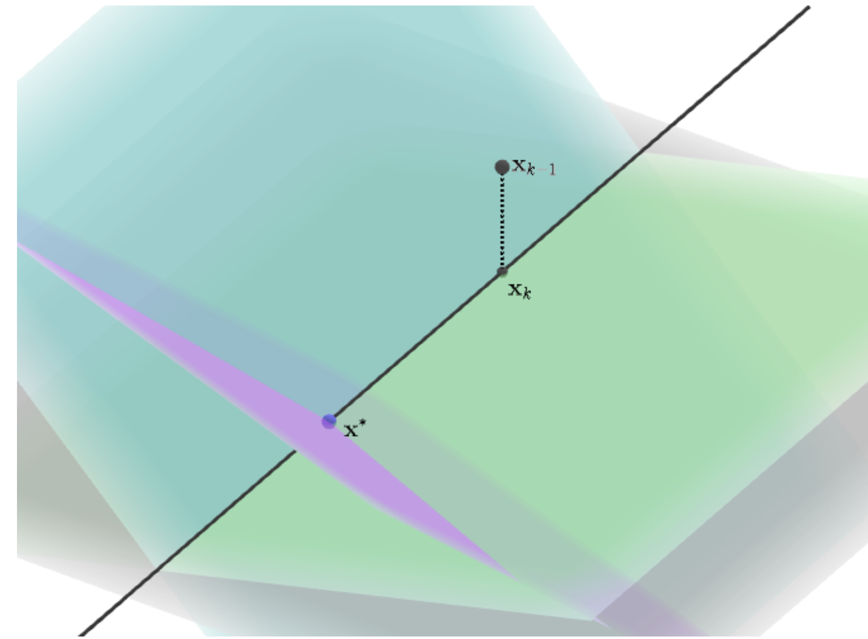

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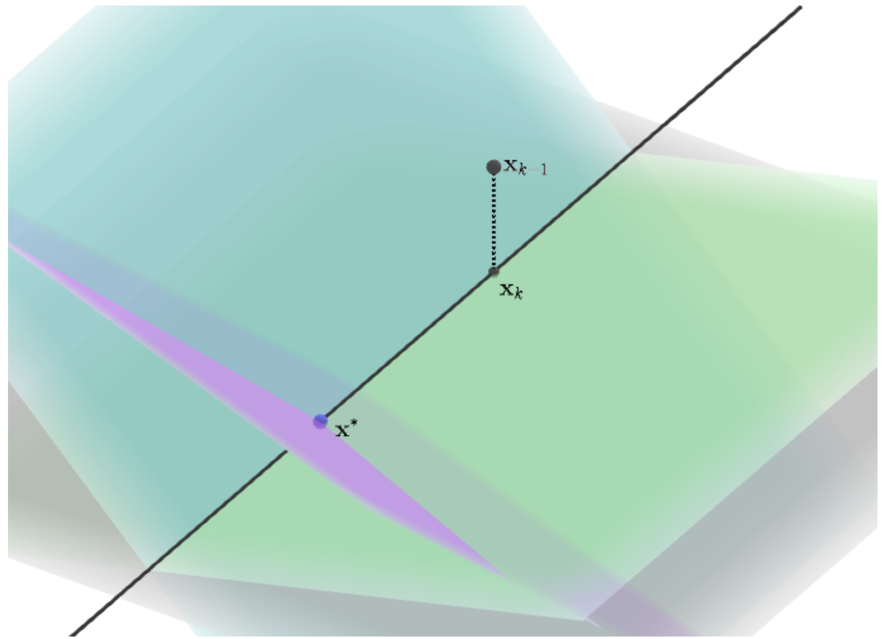

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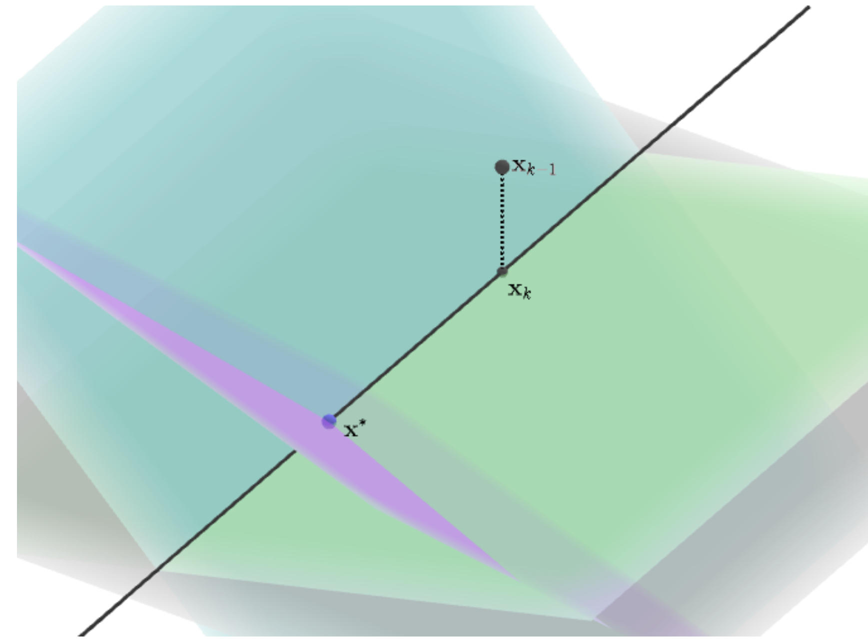
- Gauss-Seidel methods



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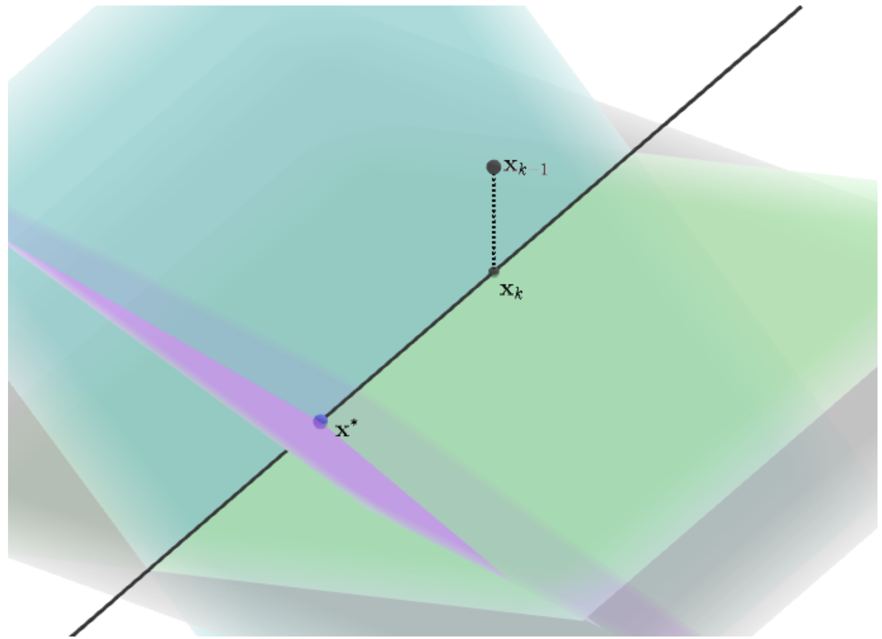
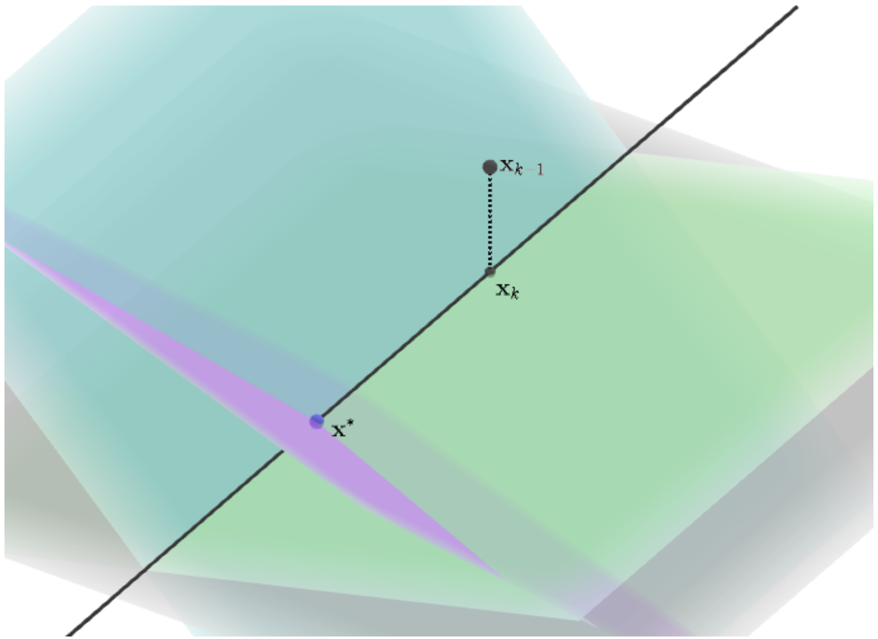
- Gauss-Seidel methods
- coordinate descent methods



Example: Block Kaczmarz Method

Given linear system measurement matrix A and measurement vector $\mathbf{b}$, initial iterate $\mathbf{x}_0$, and sets of row indices $T = \{\tau_1, \cdots, \tau_d\}$, for $k = 1, 2, \cdots$:

- Choose row block τ uniformly at random from T.
- $\mathbf{x}_k = \mathbf{x}_{k-1} + A_\tau^\dagger(\mathbf{b}_\tau - A_\tau \mathbf{x}_{k-1})$



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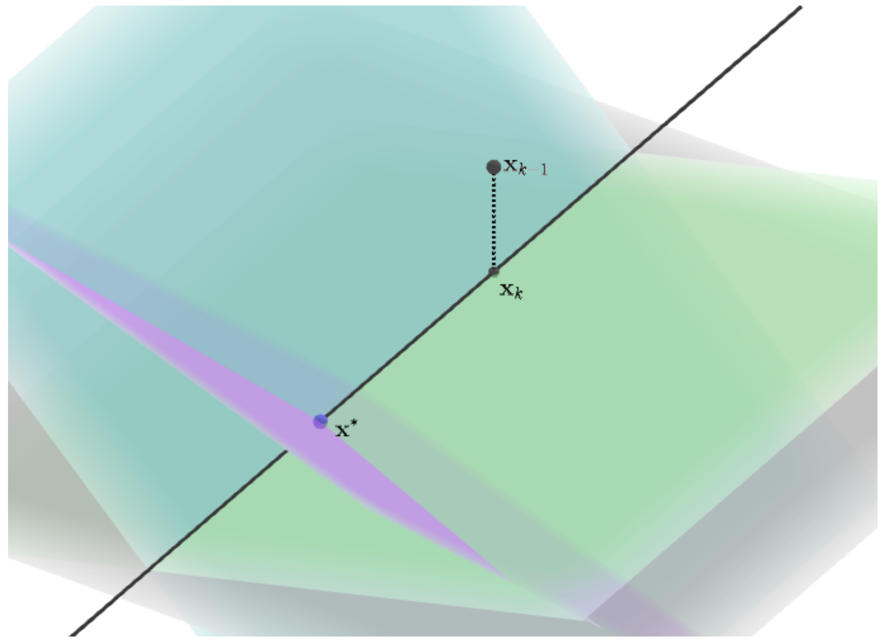

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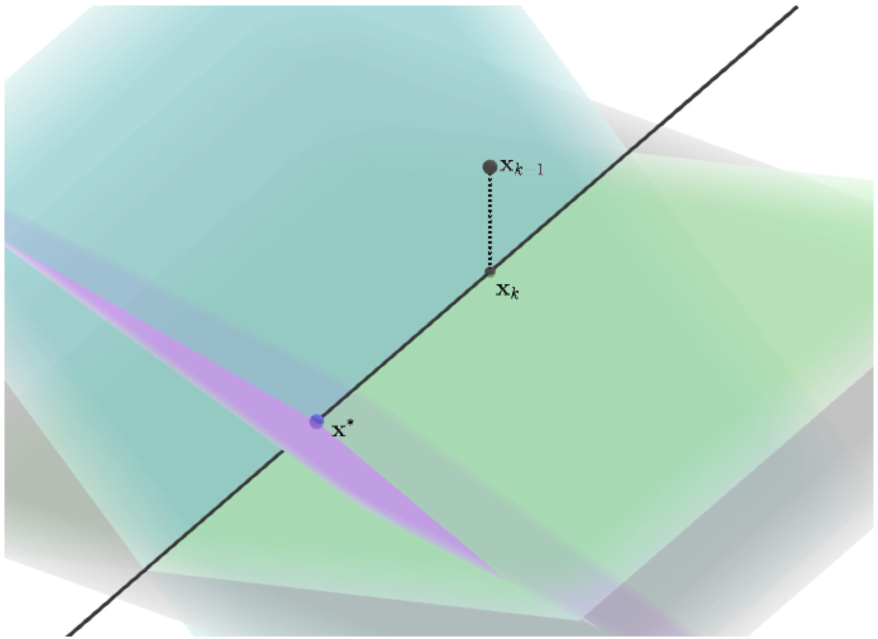

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How to choose the subset of rows, T?

Definition: A (d, α, β) **row paving** of a matrix $\mathbf{A}$ is a partition $T = \{\tau_1, \tau_2, \cdots, \tau_d\}$ of the row indices that satisfies

$$\alpha \leq \lambda_{\min}(\mathbf{A}_\tau \mathbf{A}_\tau^\top) \text{ and } \lambda_{\max}(\mathbf{A}_\tau \mathbf{A}_\tau^\top) \leq \beta \text{ for each } \tau \in T. \text{ [1]}$$

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where r and R are the minimum and maximum, respectively, number of blocks in which a single row appears, i.e., $r = \min_{i \in [m]} |\{\tau_l \in T : i \in \tau_l\}|$ and $R = \max_{i \in [m]} |\{\tau_l \in T : i \in \tau_l\}|$.

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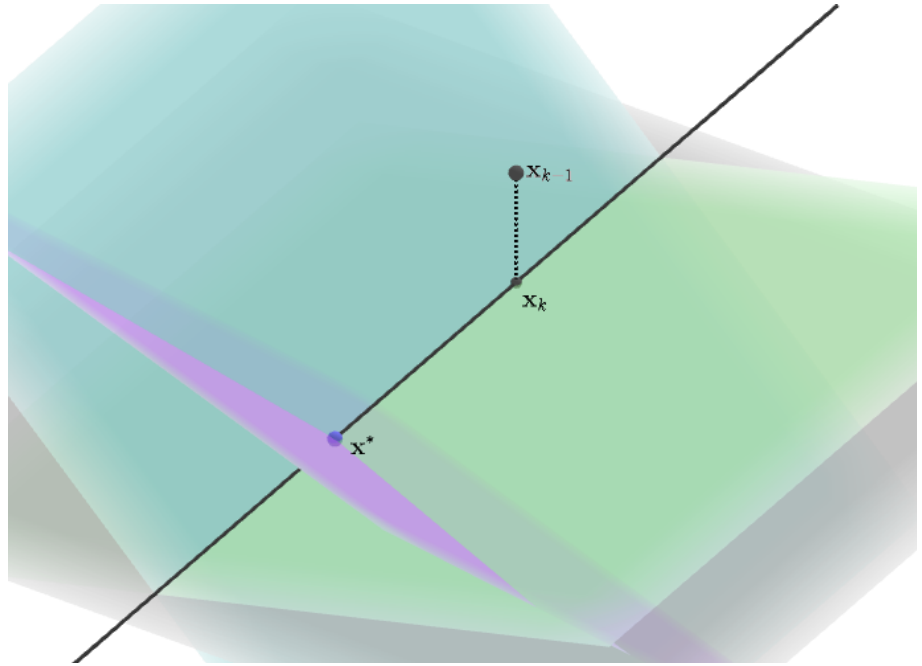

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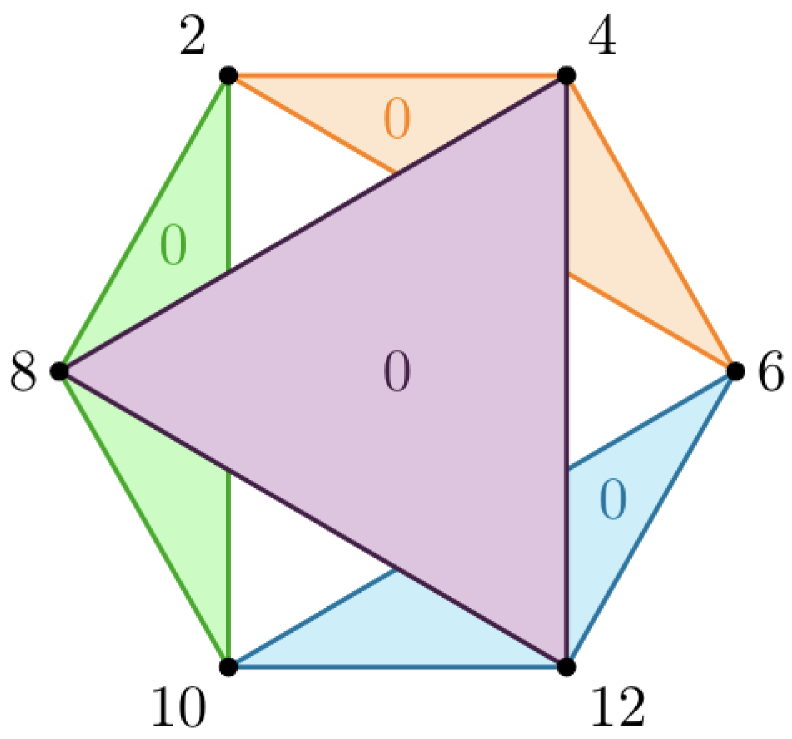

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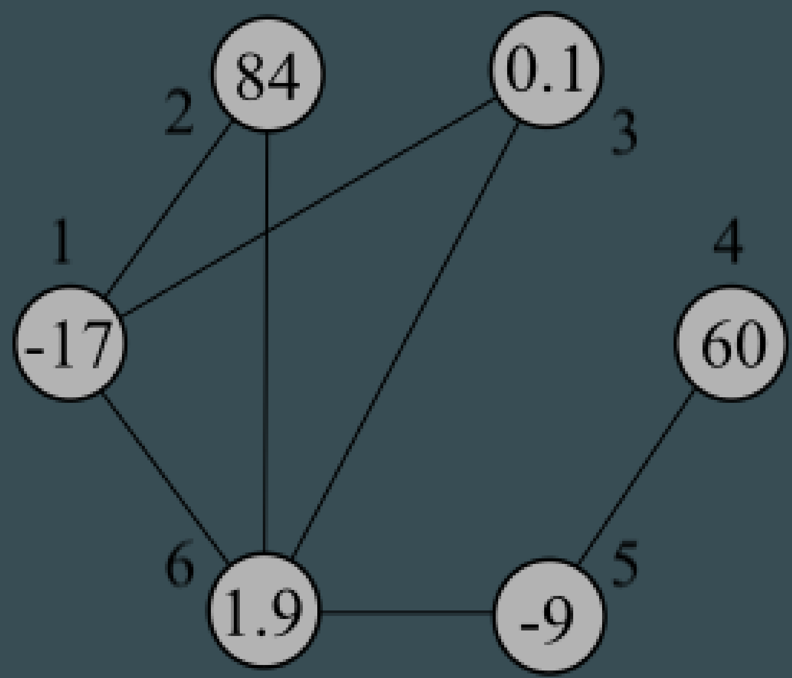

- formulate averaging consensus as a **homogenous linear system** (e.g., Laplacian system, incidence system)

Loizou, N., & Richtárik, P. (2021). Revisiting randomized gossip algorithms: General framework, convergence rates and novel block and accelerated protocols. IEEE Transactions on Information Theory, 67(12), 8300-8324.

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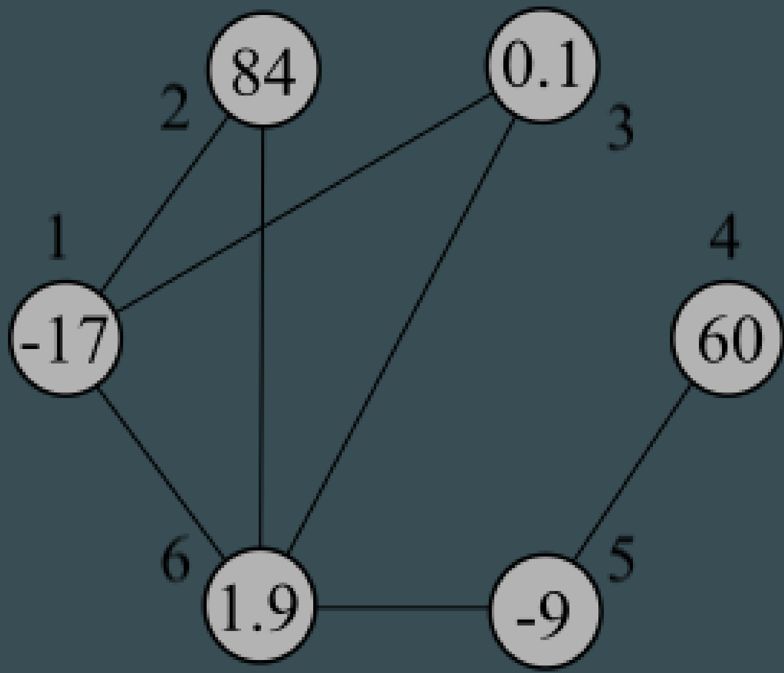

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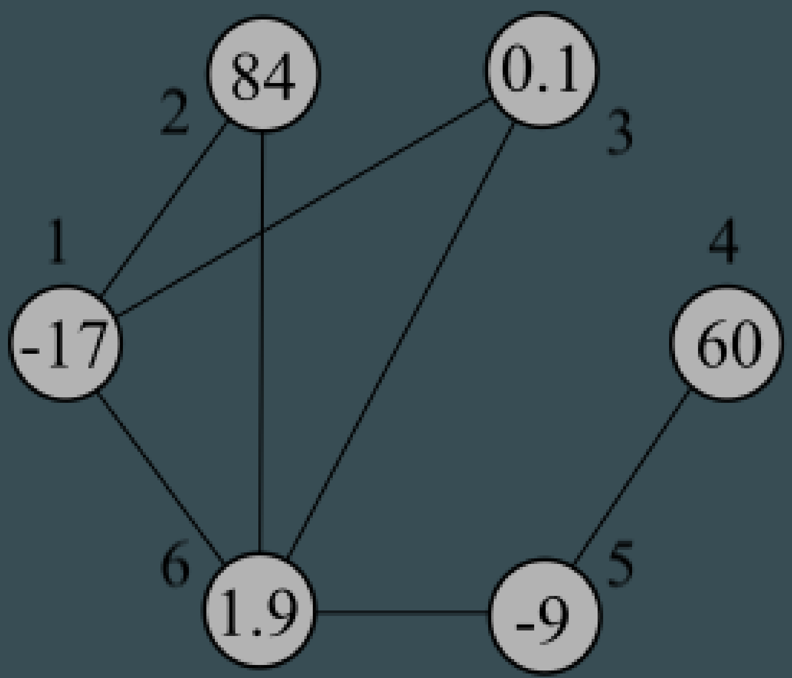

- formulate averaging consensus as a **homogenous linear system** (e.g., Laplacian system, incidence system)
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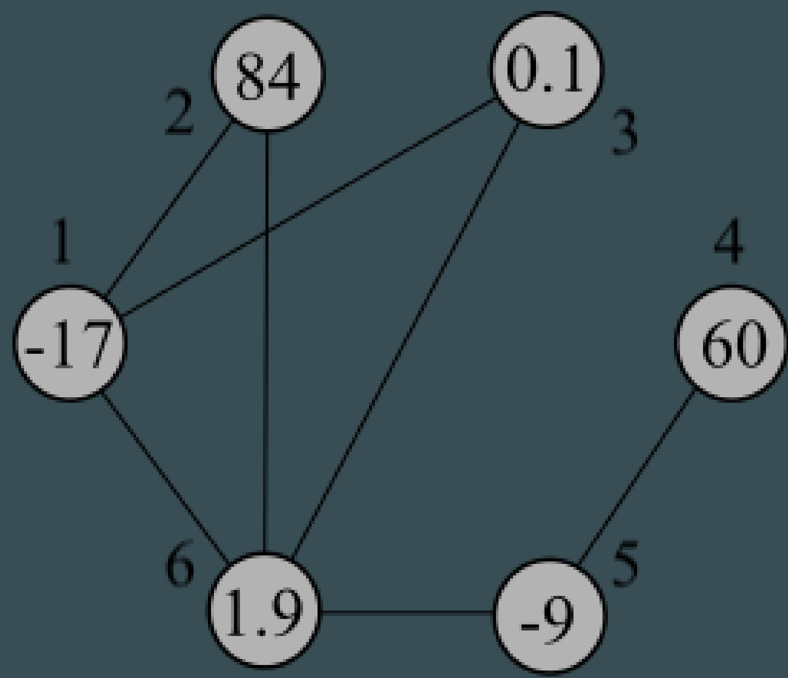

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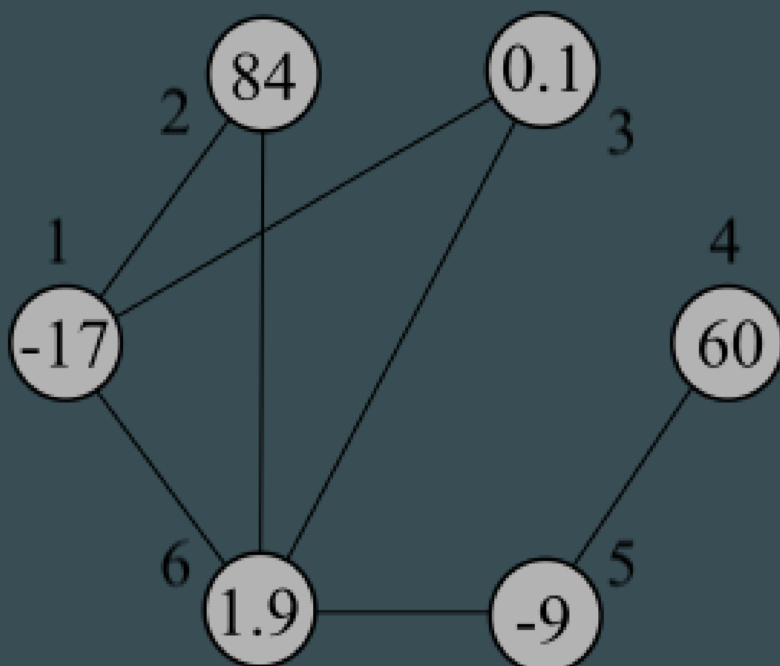

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$$\mathbf{Q} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{-1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{-1} \\ 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{-1} \\ 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{-1} \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{-1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-1} \end{bmatrix}$$

The bridge application...

The block gossip method with blocks T produces the same iterates as the block Kaczmarz method performed with $\mathbf{A} = \mathbf{Q}$, $\mathbf{b} = \mathbf{0}$, and $\mathbf{x}_0 = \mathbf{c}_0$ with row blocks corresponding to the same edge sets as T.

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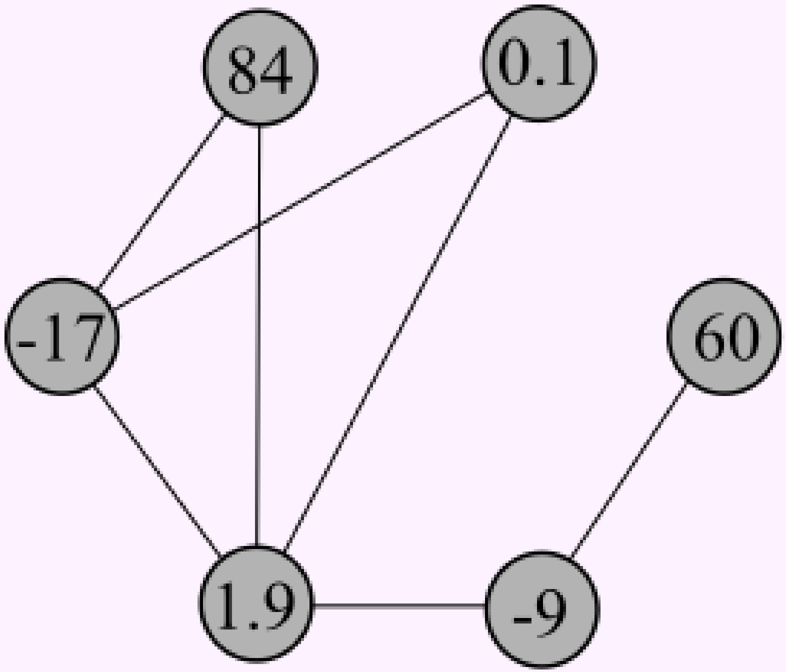

Block Kaczmarz Convergence

Theorem: Consider the least-squares problem $\mathbf{min}\,\|\mathbf{Ax} - \mathbf{b}\|_2^2$ where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is not necessarily full-rank and $\mathbf{b} \in \mathbb{R}^m$. Let $T = \{\tau_1, \cdots, \tau_d\}$ be a (d, α, β, r, R) covering (not necessarily a paving) of the rows of $\mathbf{A}$. Let $\mathbf{x}_j$ denote the jth iterate produced by Block RK on the system defined by $\mathbf{A}$ and $\mathbf{b}$ with initial iterate $\mathbf{x}_0$, let $\mathbf{x}^* := \mathbf{argmin}_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|_2^2$, and let $\mathbf{e} := \mathbf{Ax}^* - \mathbf{b}$. Then we have

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These generalizations are important for application to average consensus and block gossip methods, but are likely of interest in other applications.

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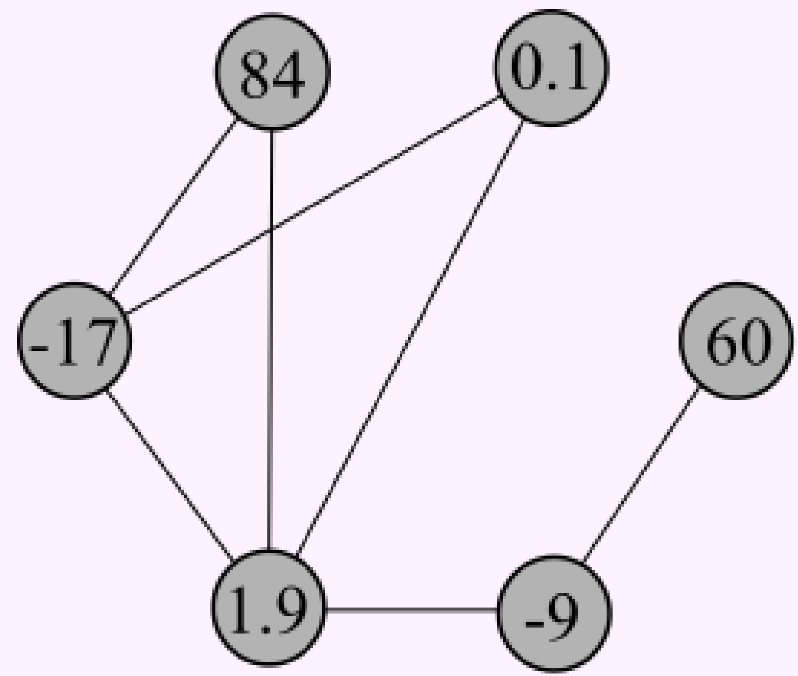
The Block Kaczmarz convergence result yields as a corollary a **convergence result for the block gossip method**.



Block Gossip Convergence

Corollary: Suppose graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is connected, $\mathbf{Q} \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{V}|}$ is the incidence matrix for $\mathcal{G}$, and $T = \{\tau_1, \cdots, \tau_d\}$ is a (d, α, β, r, R) row covering for $\mathbf{Q}$ with $M = \max_{i \in [d]} |\tau_i|$. Then the block gossip method with blocks determined by T converges at least linearly in expectation with the guarantee

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- If T consists of independent edge sets, the rate constant can be bounded by $\left(1 - \frac{r\alpha(\mathcal{G})}{2d}\right)$.
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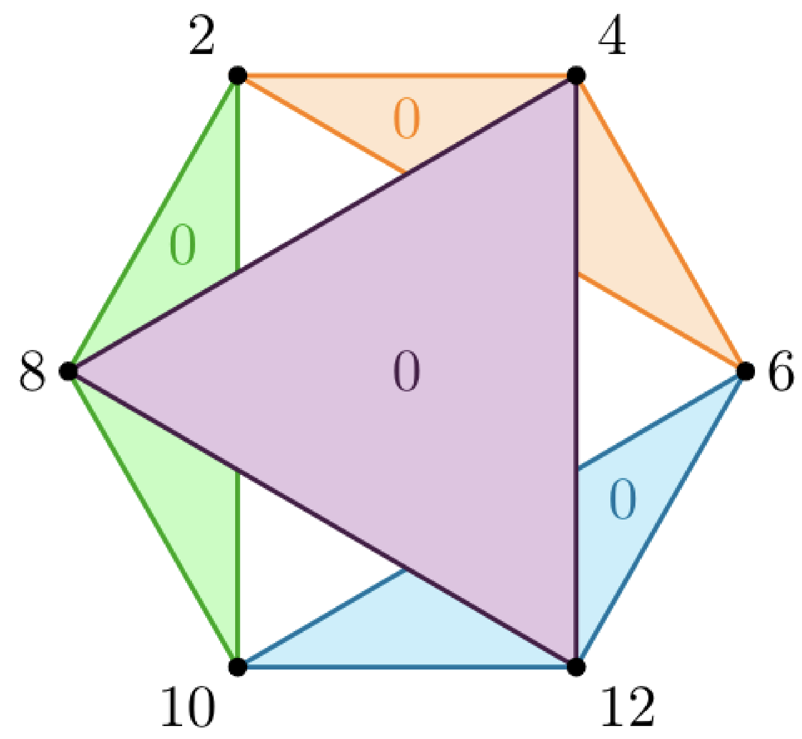

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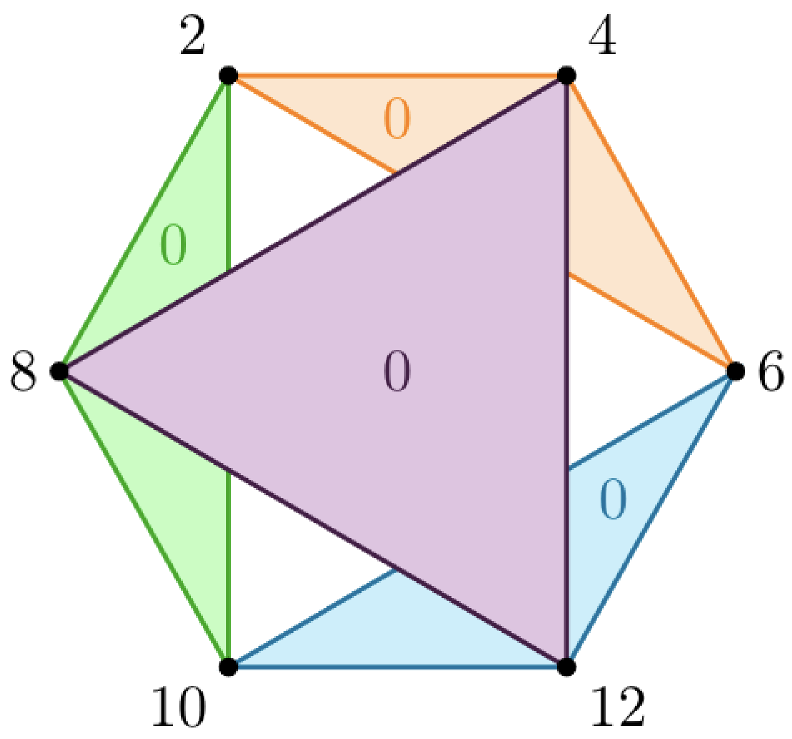

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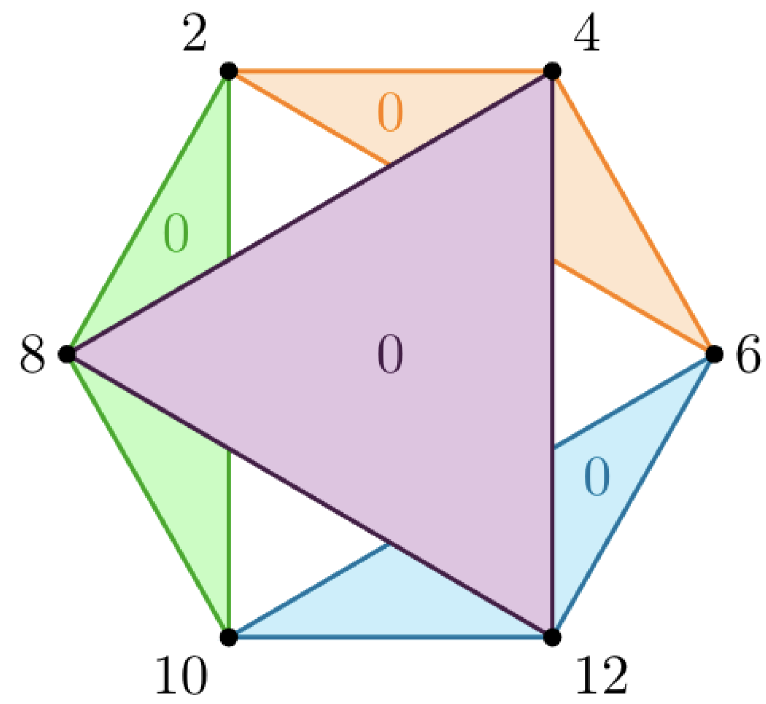
There is a natural bridge between many problems regarding consensus dynamics on networks and classical iterative methods from numerical linear algebra.



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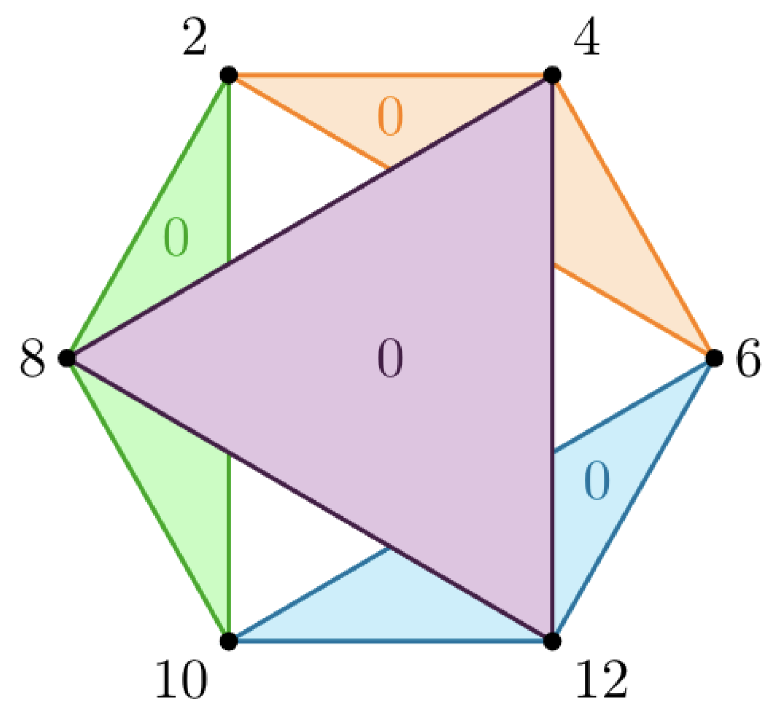

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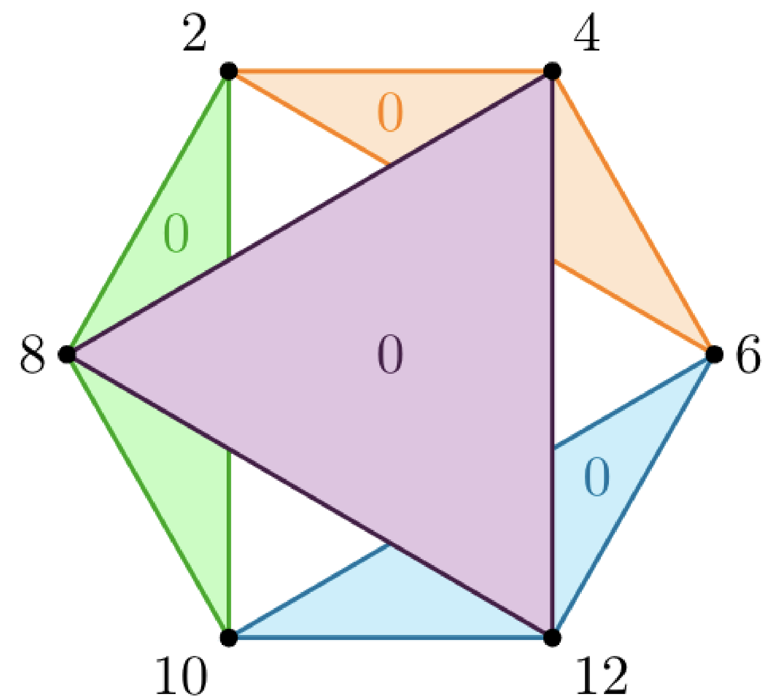

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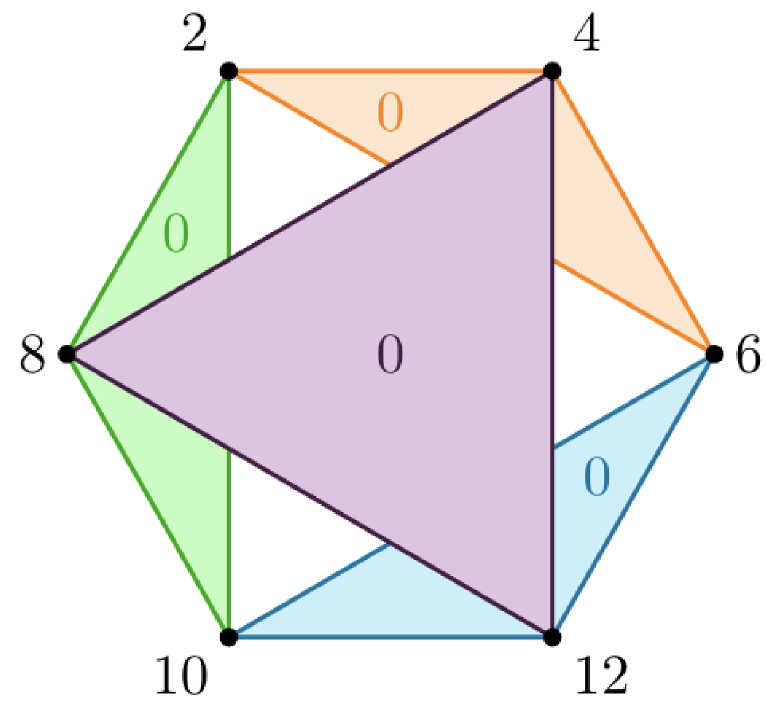

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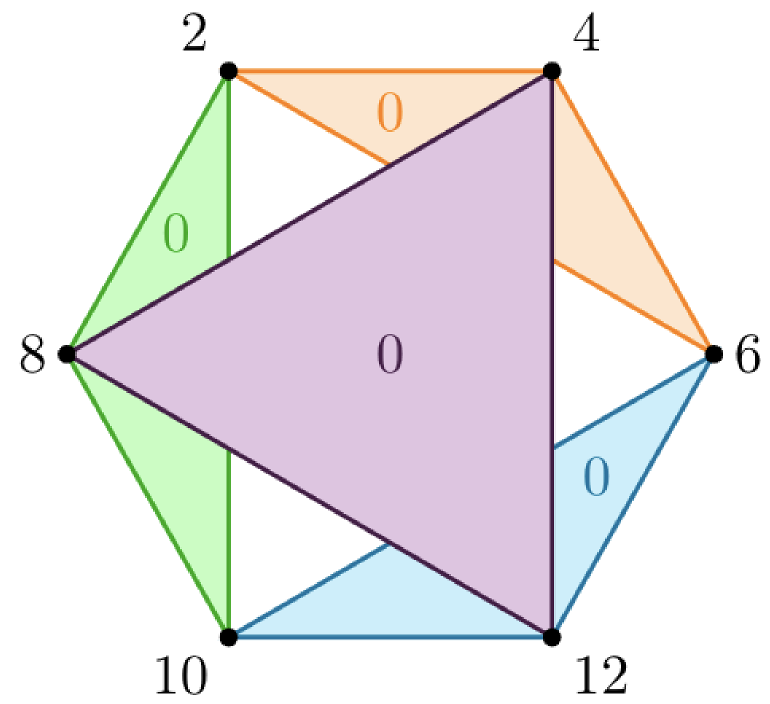

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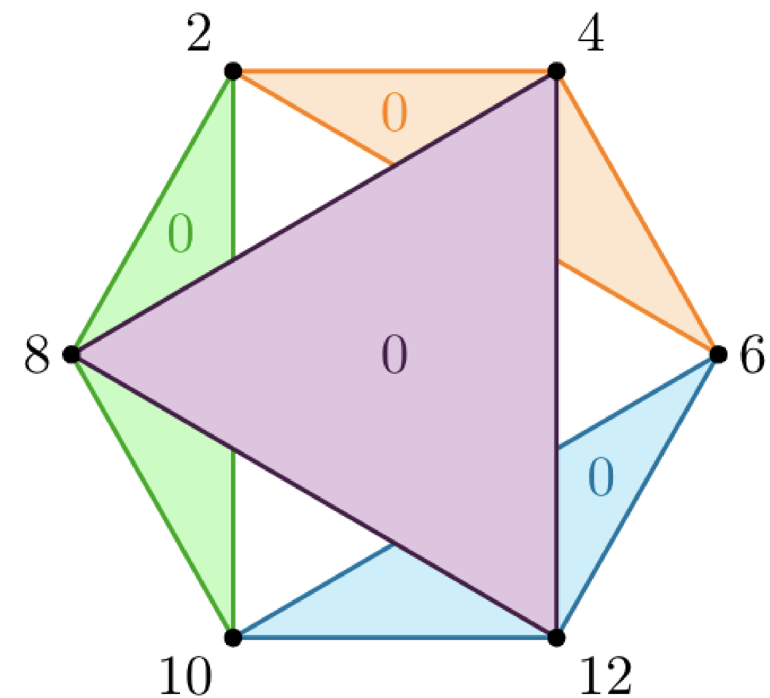
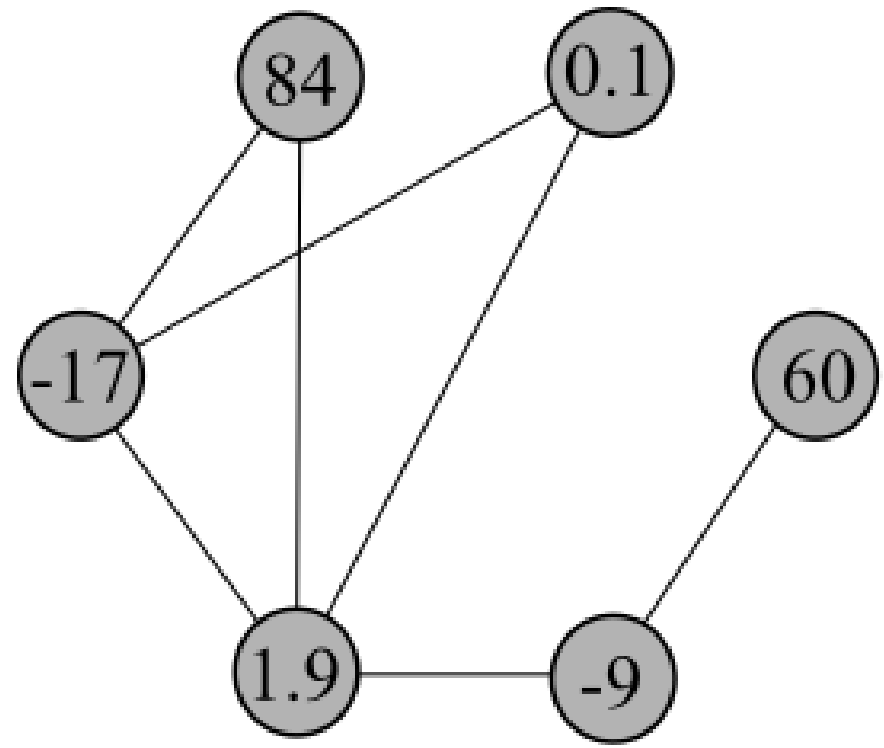
To tackle more complex models (e.g., bounded confidence, imperfect communication, etc.) we can look to the ever-growing body of NLA literature on variants of iterative methods.



Current Work

Show that the unbounded **Hegselmann-Krause (HK) model** can be analyzed under the **Jacobi and Gauss-Seidel method** framework.

Hector Tierno
HMC



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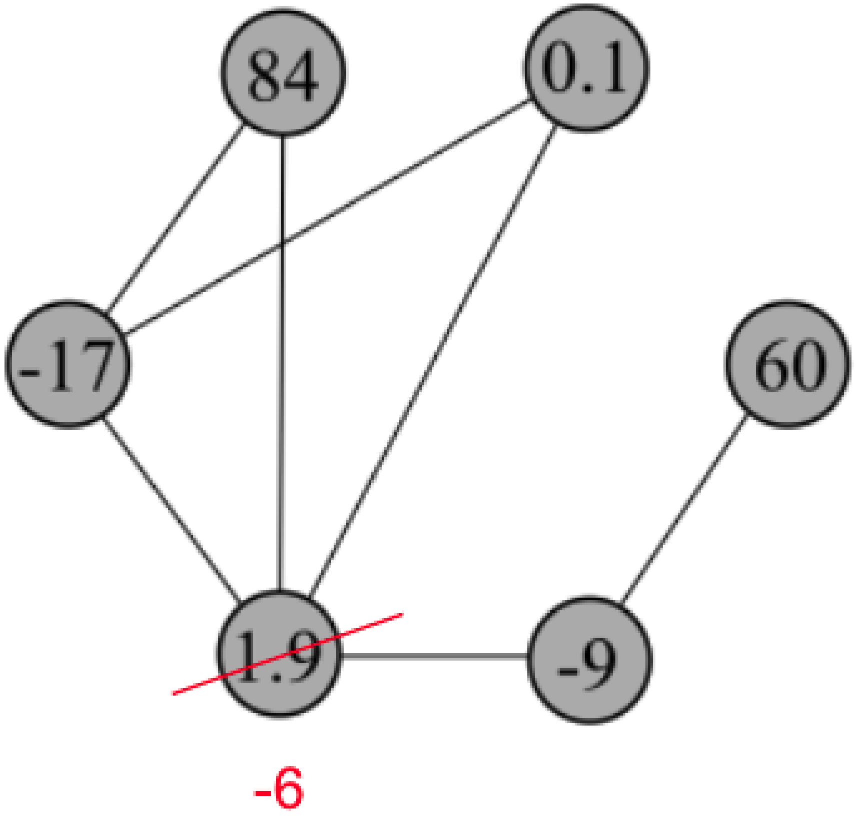
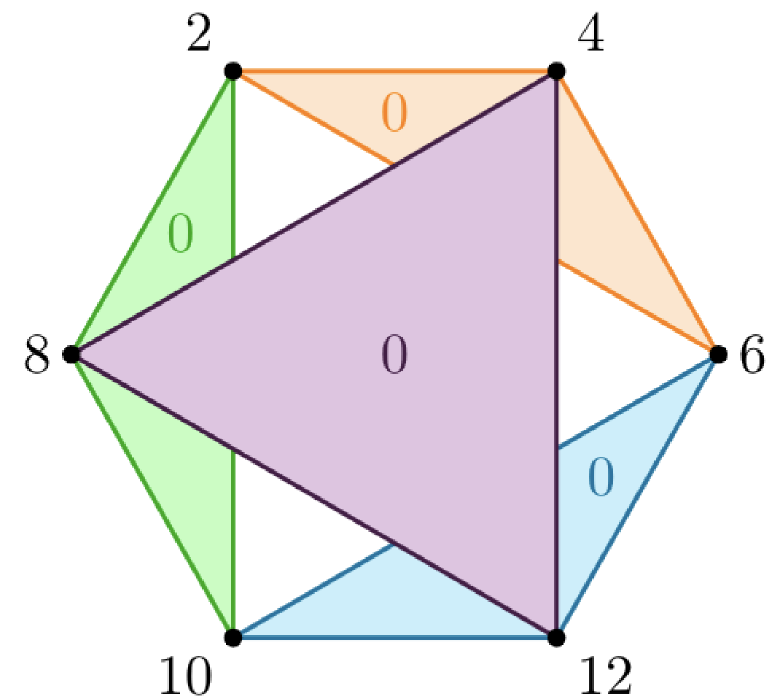


Future Work

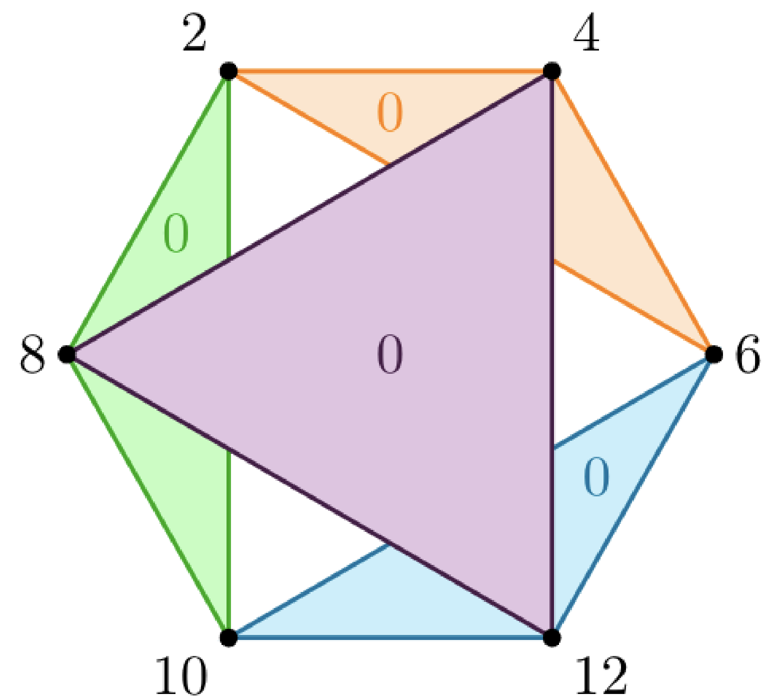
Analyze **bounded** models through the framework of residual-constrained iterative methods.



Future Work

Analyze **bounded** models through the framework of residual-constrained iterative methods.

Understand limit of consensus models via NLA and algebraic graph theory literature.



Meng, X. F., Van Gorder, R. A., & Porter, M. A. (2018). Opinion formation and distribution in a bounded-confidence model on various networks. Physical Review E, 97(2), 022312.

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Extend work to models on **hypergraphs**.

Hickok, A., Kureh, Y., Brooks, H. Z., Feng, M., & Porter, M. A. (2022). A bounded-confidence model of opinion dynamics on hypergraphs. SIAM Journal on Applied Dynamical Systems, 21(1), 1-32.

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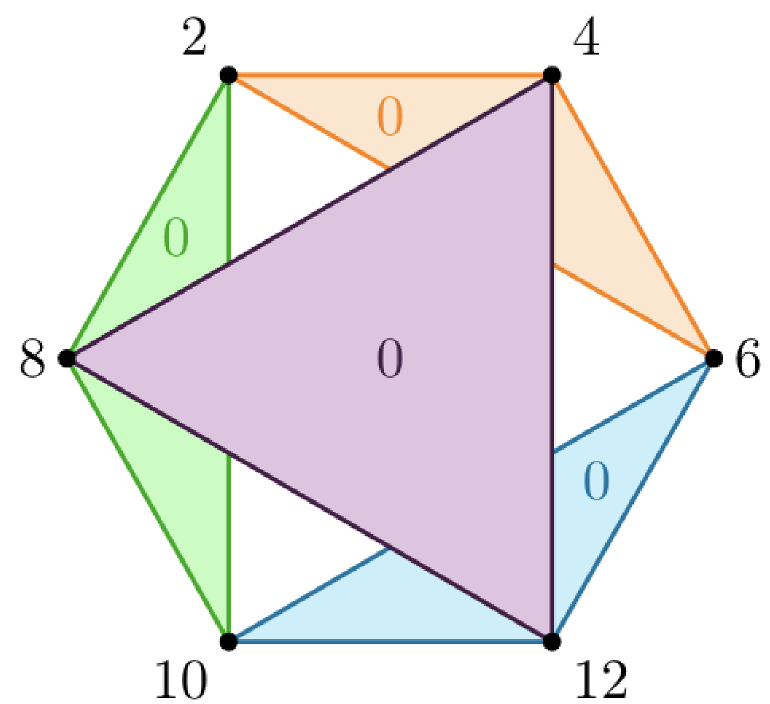

Summary

The **average consensus problem** may be formulated as a least-squares problem.

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Chen Yap
Planet Labs Inc.

JH, Benjamin Jarman, and Chen Yap (2022). Paving the Way for Consensus: Convergence of Block Gossip Algorithms. *Submitted*.

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Summary

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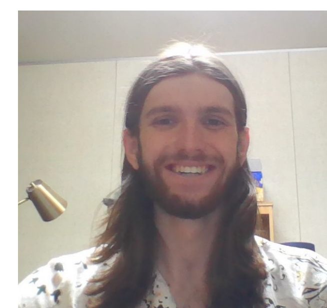

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The **average consensus problem** may be formulated as a least-squares problem.

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This technique may be exploited for **other models of consensus dynamics on networks**.

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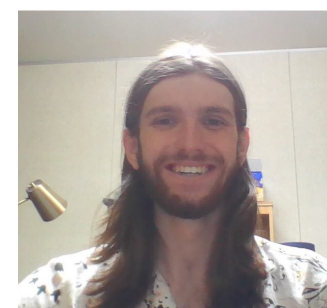

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HMC

Thanks everyone!

Questions?