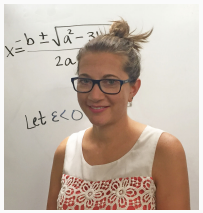
Greedy and Randomized Projection Methods

Jamie Haddock

UCLA CAM Colloquium,

October 30, 2019

Computational and Applied Mathematics
UCLA

joint with Jesús A. De Loera, Deanna Needell, and Anna Ma
https://arxiv.org/abs/1802.03126 (BIT Numerical Mathematics 2019)
https://arxiv.org/abs/1605.01418 (SISC 2017)

BIG Data

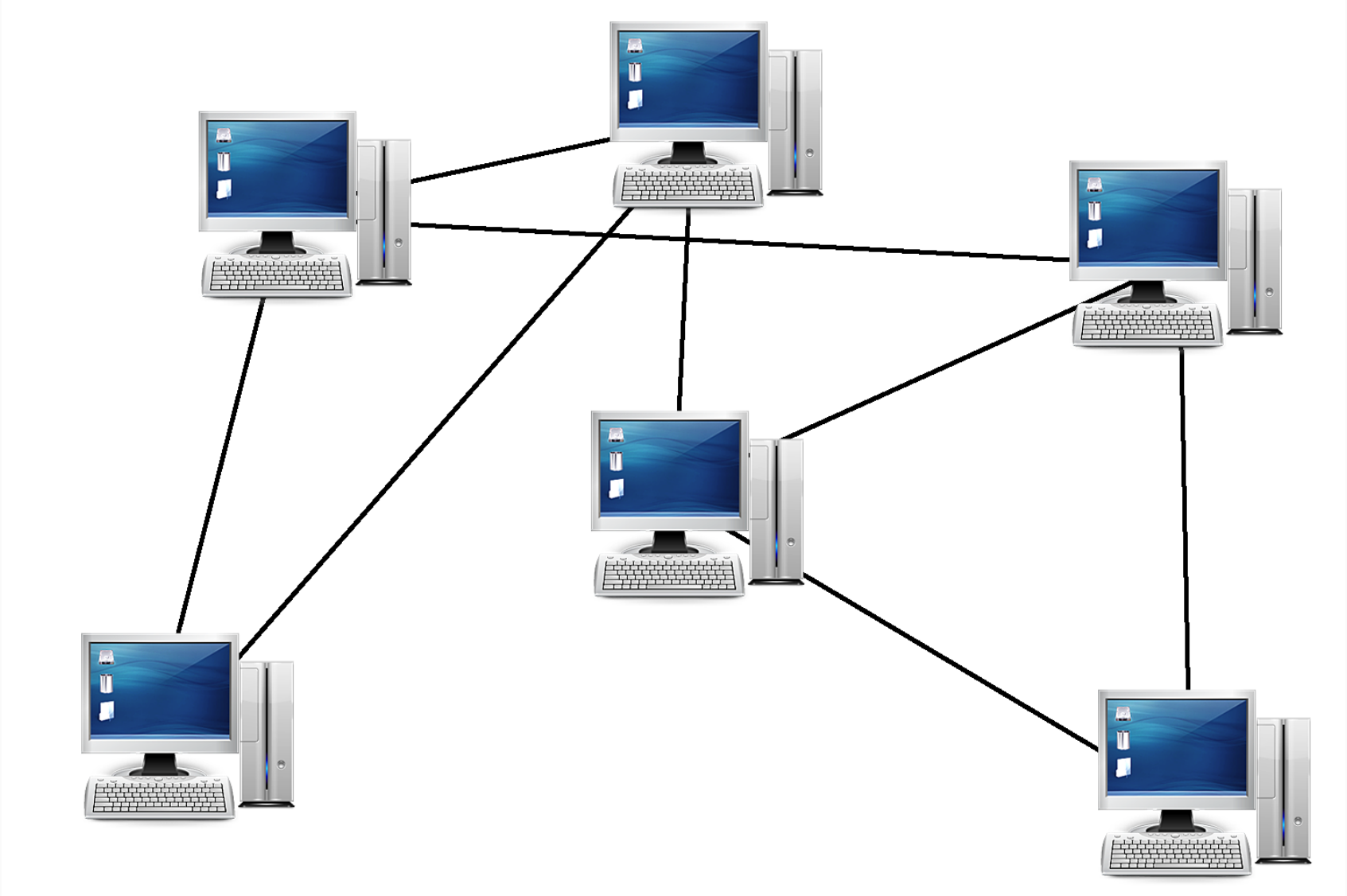
The big data opportunity

DellWorld'15

BIG Data


The Economist
Obama the warrior
Misgoverning Argentina
The economic shift from West to East
Genetically modified crops blossom
The right to eat cats and dogs
FEBRUARY 27TH–MARCH 5TH 2010
Economist.com
The data deluge
AND HOW TO HANDLE IT: A 14-PAGE SPECIAL REPORT

Motivation: Average Consensus



[0] By Everaldo Coelho (YellowIcon); - All Crystal icons were posted by the author as LGPL on kde-look;, LGPL, https://commons.wikimedia.org/w/index.php?curid=7288951.

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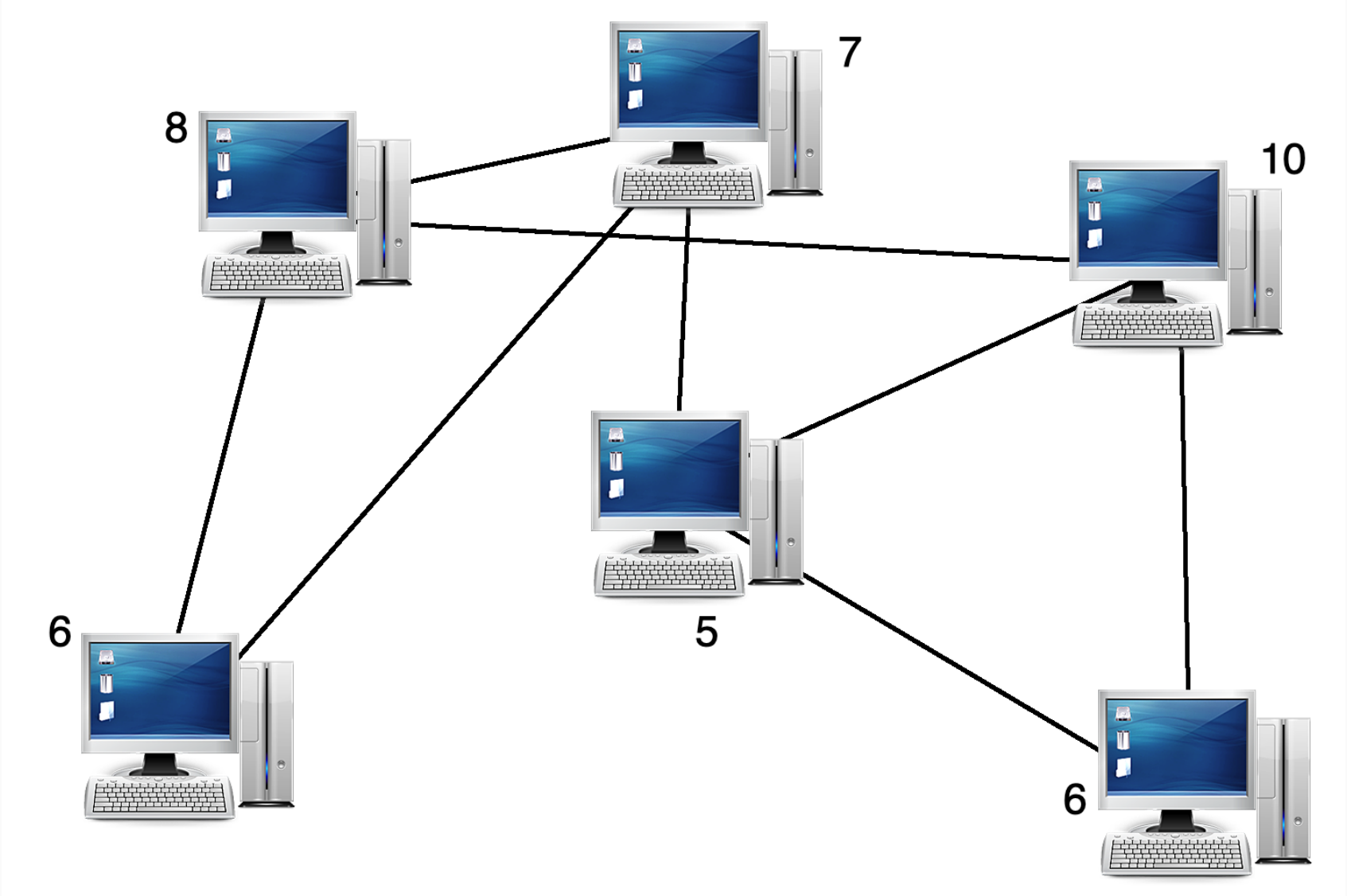

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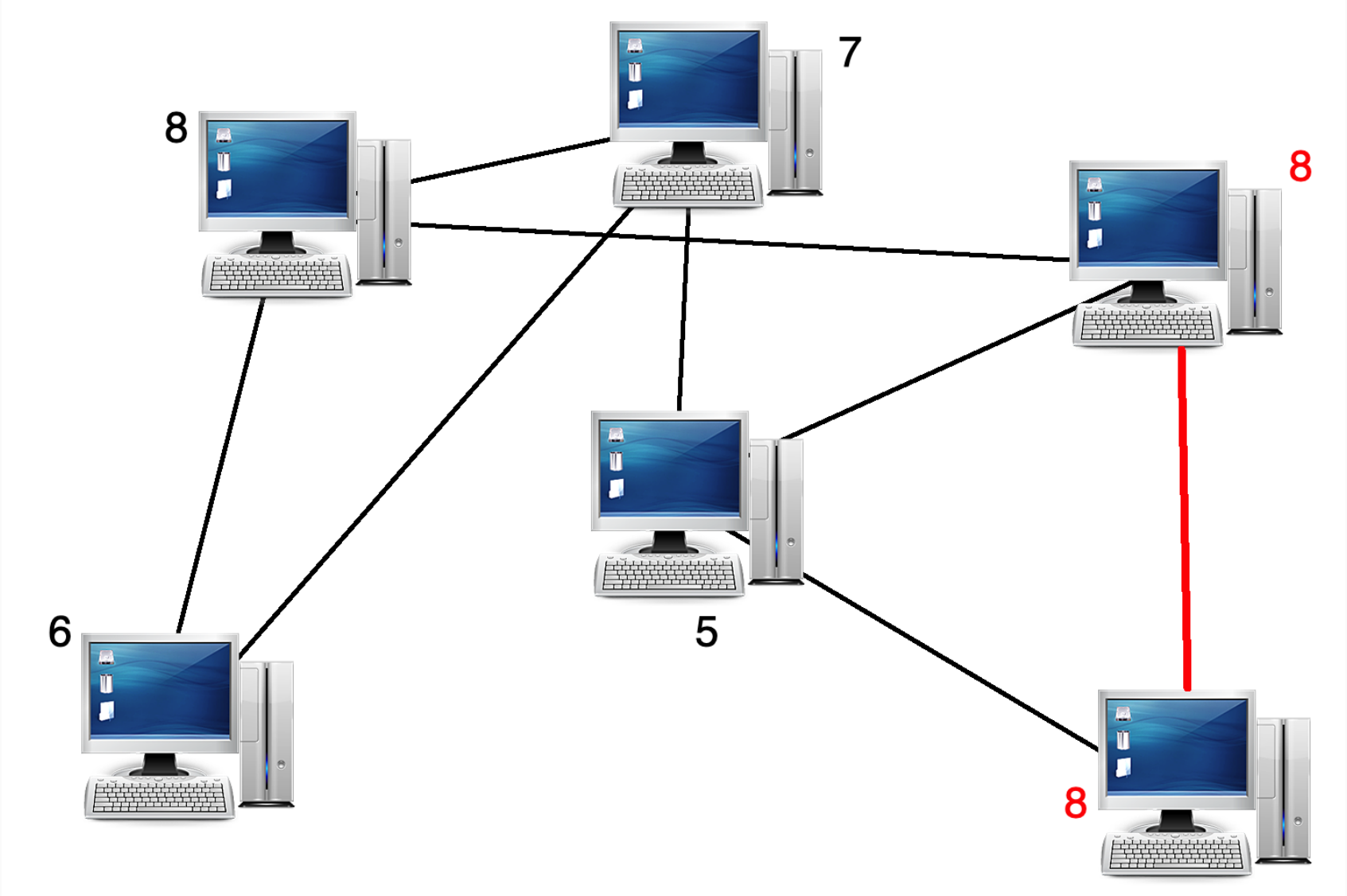

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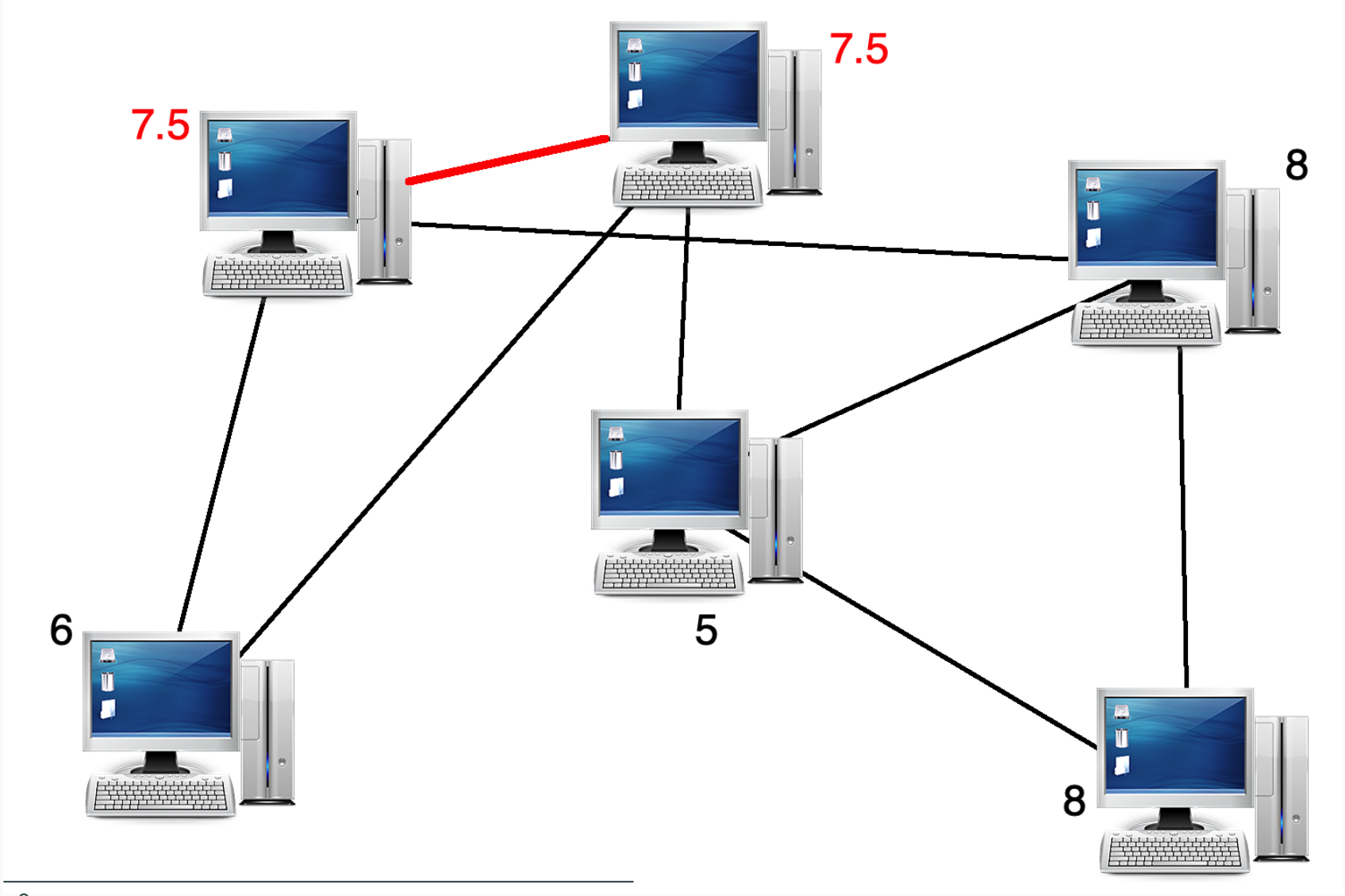

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AC solution:

$$\text{argmin}_{\mathbf{x}} \frac{1}{2}\|\mathbf{x}-\mathbf{c}\|^2 \text{ s.t. } \begin{bmatrix} 1 & 0 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 & \dots & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

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Applications:

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- ▷ clock synchronization
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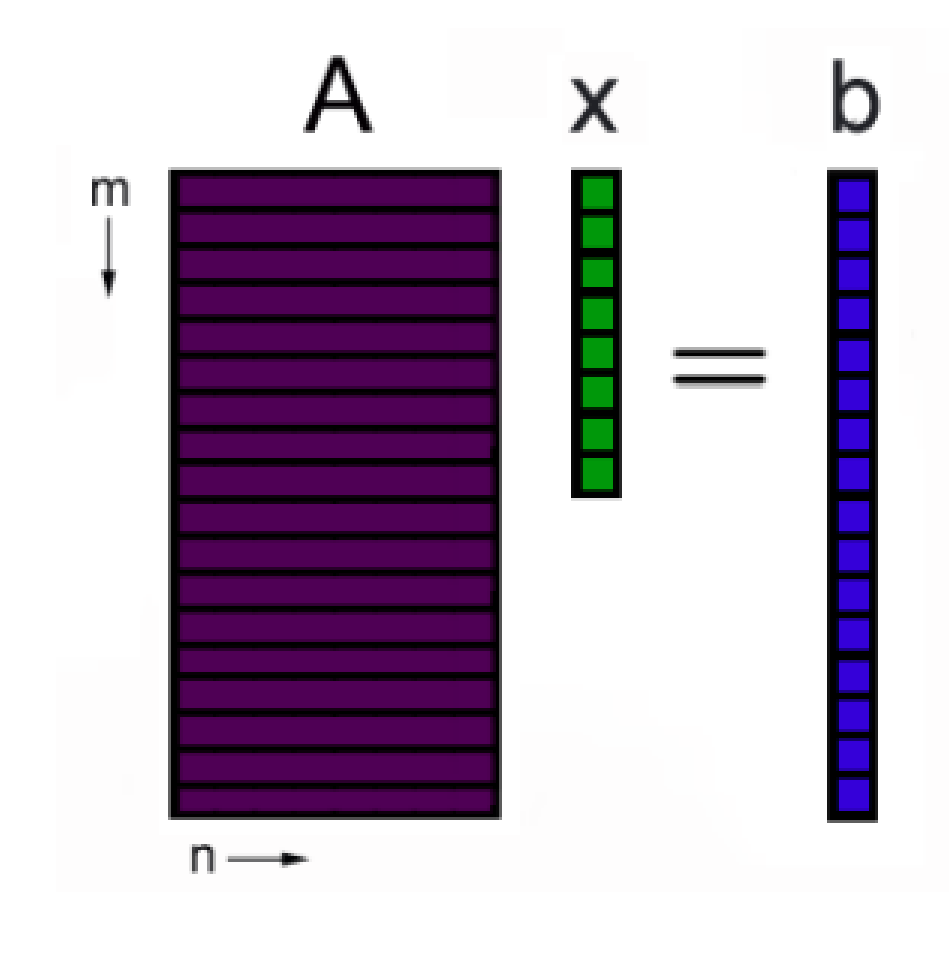
- ▷ opinion formation
- ▷ blockchain technology

General Setup

We are interested in solving highly overdetermined systems of equations (or inequalities), $A\mathbf{x} = \mathbf{b}$ ($A\mathbf{x} \leq \mathbf{b}$), where $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $m \gg n$. Rows are denoted $\mathbf{a}_i^T$.



Iterative Projection Methods

If $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$ is nonempty, these methods construct an approximation to a solution:

1. Randomized Kaczmarz Method

Applications:

1. Tomography (Algebraic Reconstruction Technique)

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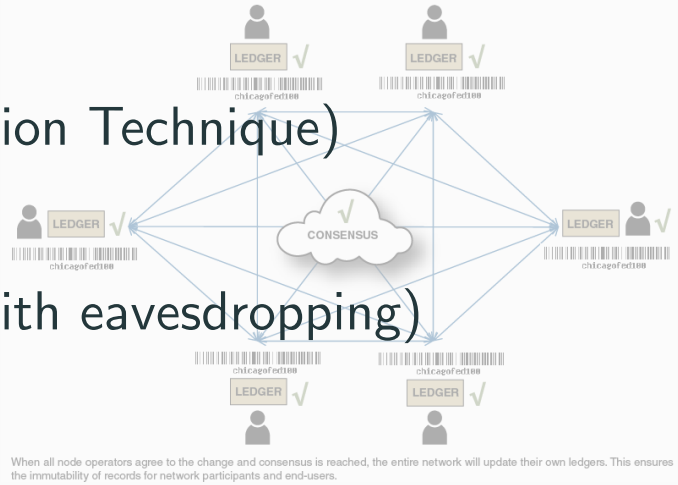
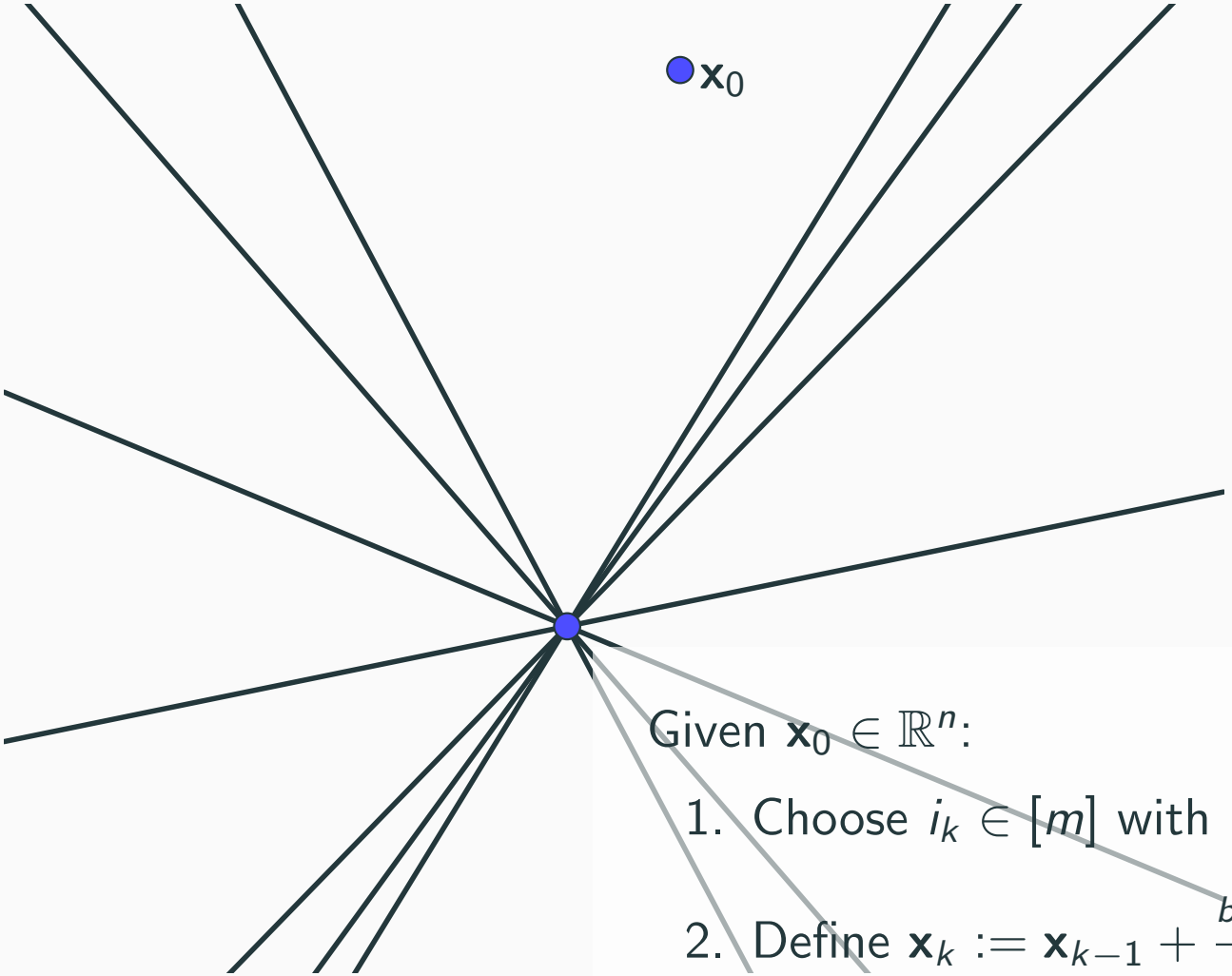
1. Randomized Kaczmarz Method
2. Motzkin's Method
3. Sampling Kaczmarz-Motzkin Methods (SKM)

Applications:

1. Tomography (Algebraic Reconstruction Technique)
2. Linear programming
3. Average consensus (greedy gossip with eavesdropping)



Kaczmarz Method



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1. Choose $i_k \in [m]$ with probability $\frac{\|\mathbf{a}_{i_k}\|^2}{\|A\|_F^2}$.
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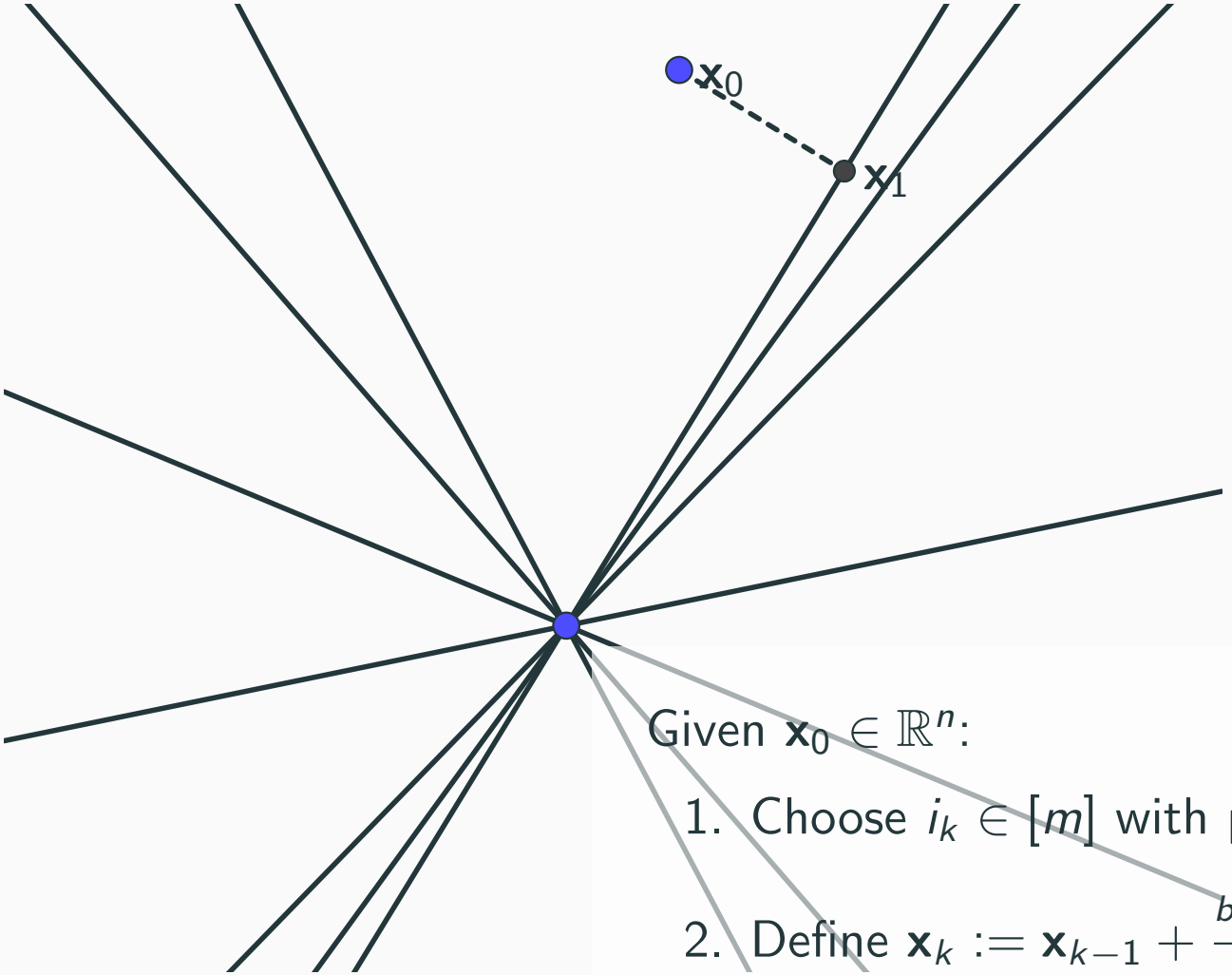

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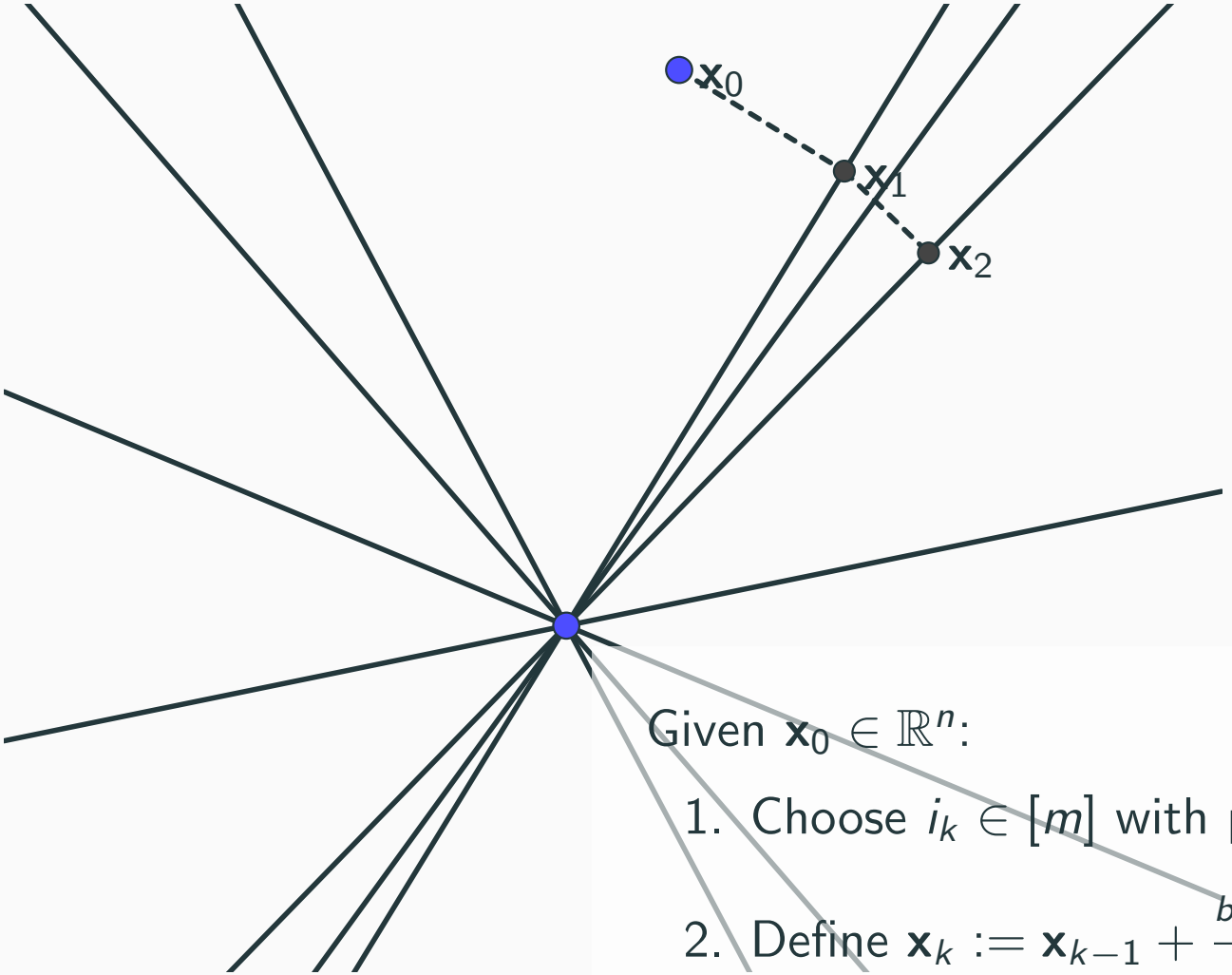

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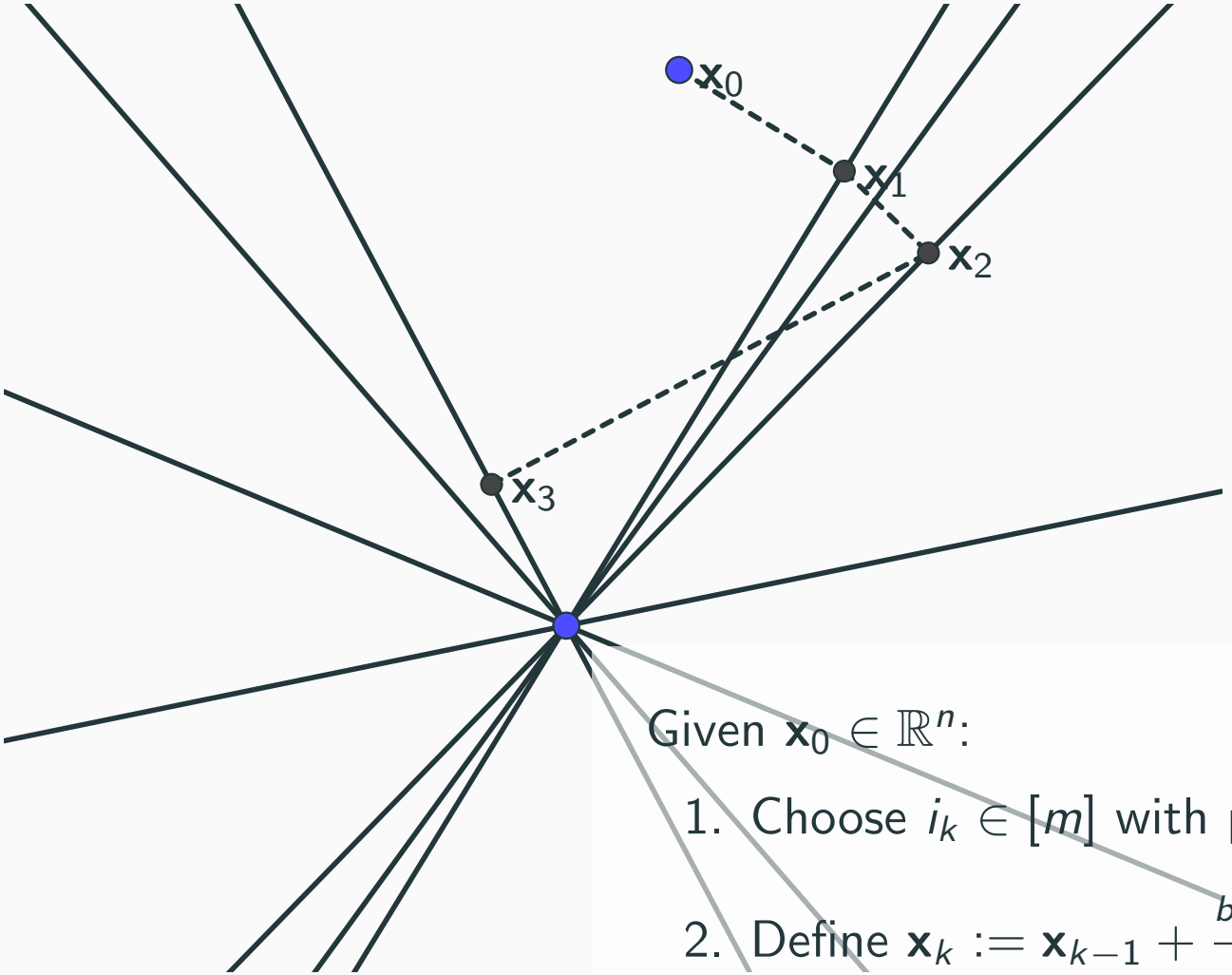

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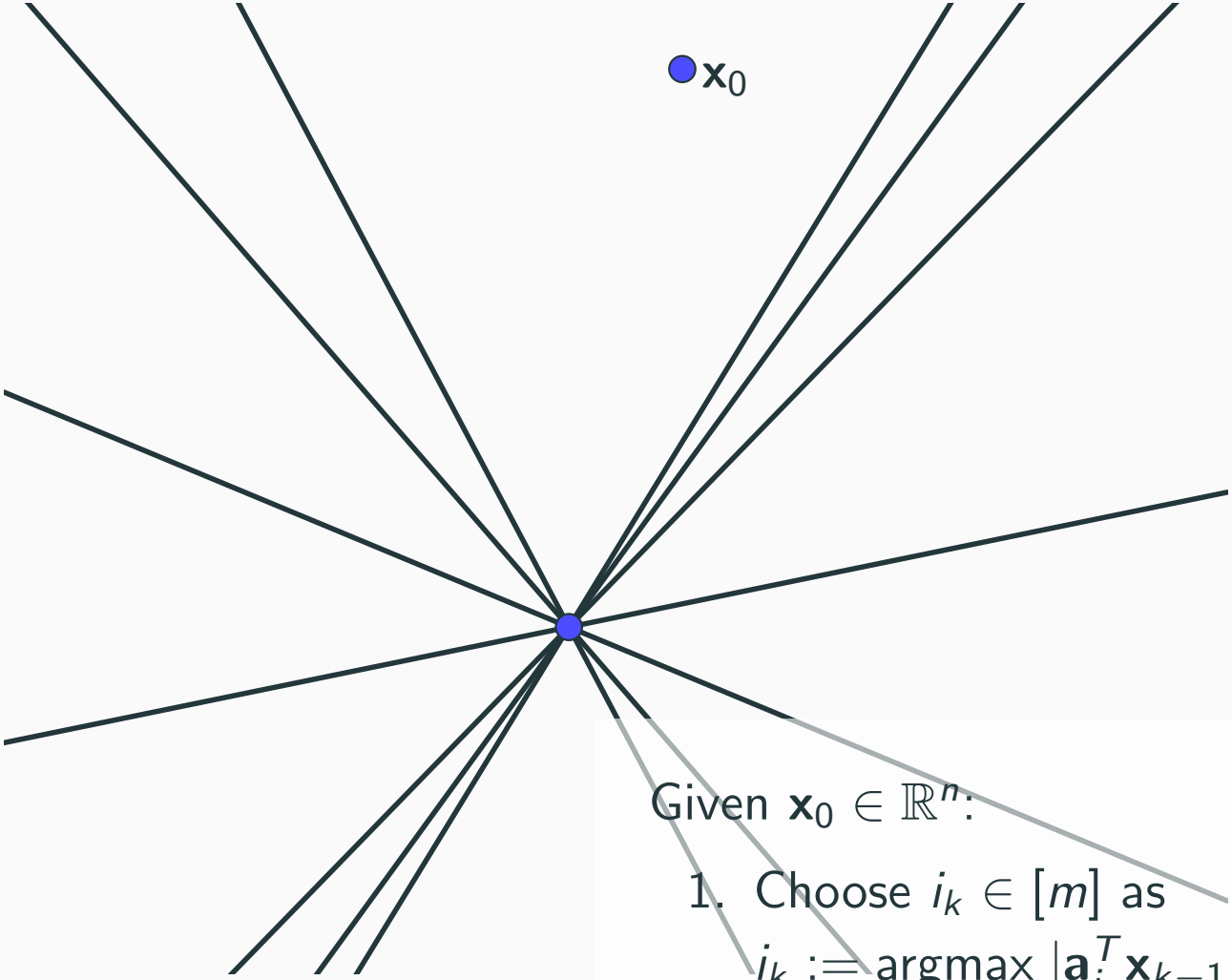

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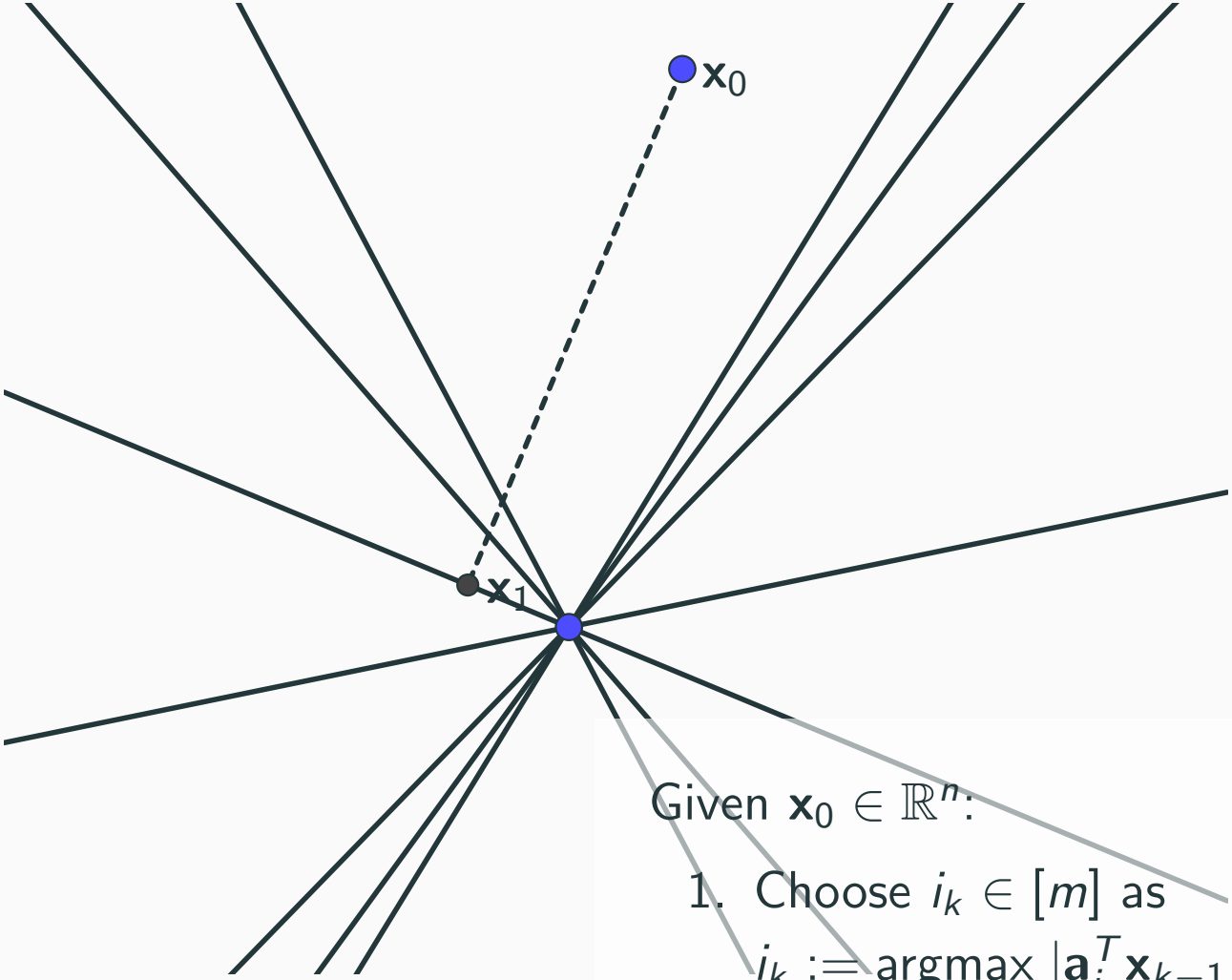

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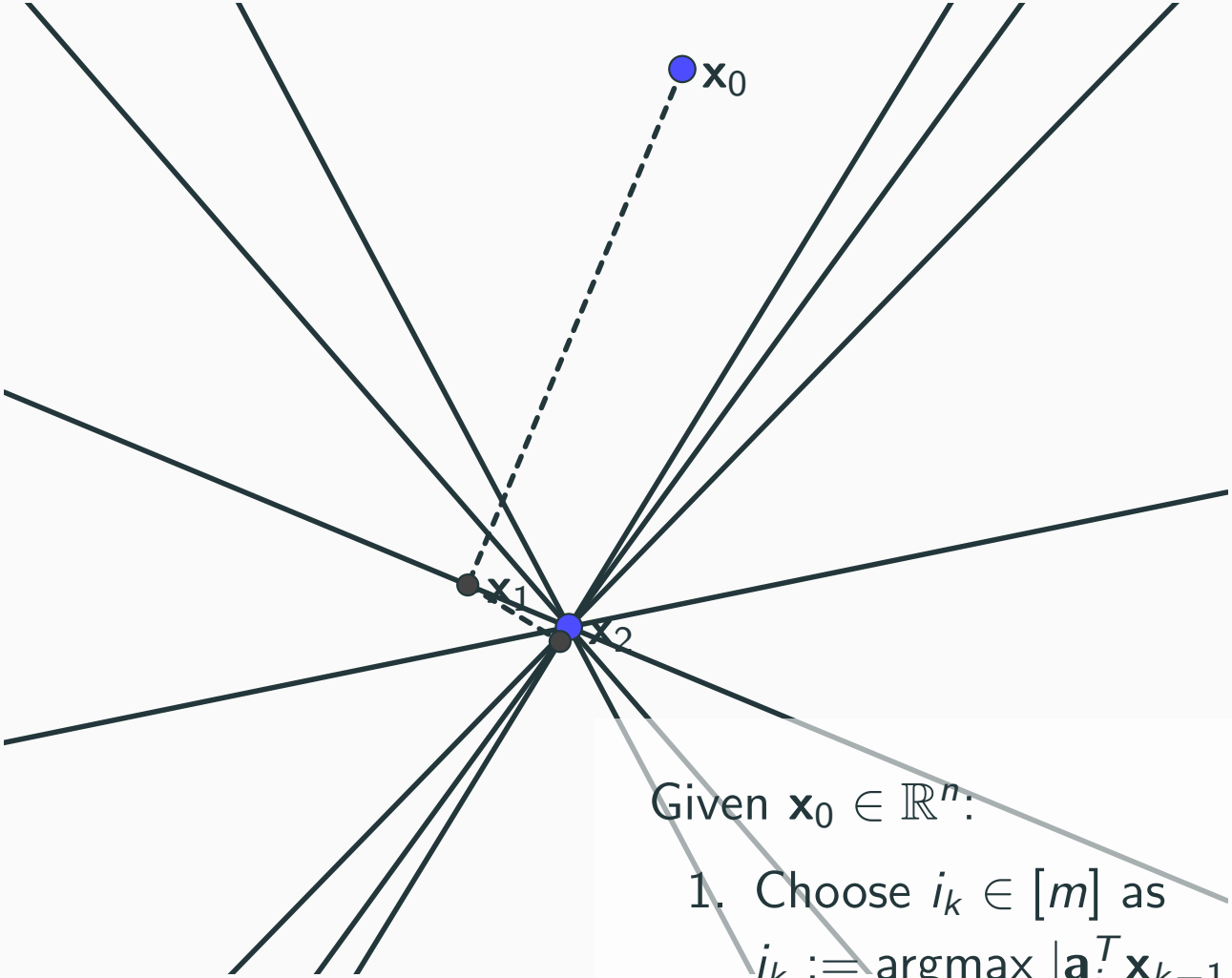

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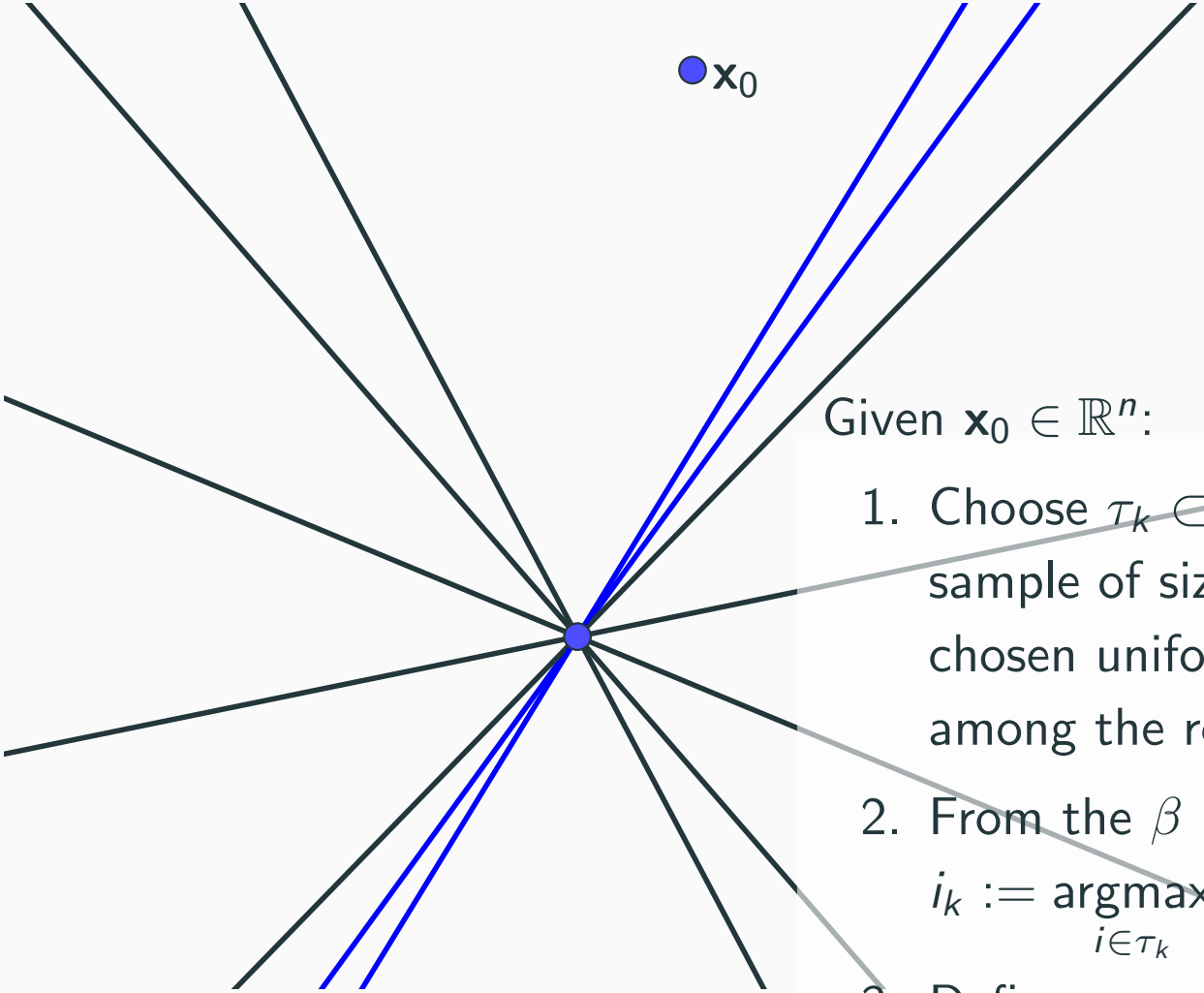
Our Hybrid Method (SKM)



Given $\mathbf{x}_0 \in \mathbb{R}^n$:

1. Choose $\tau_k \subset [m]$ to be a sample of size β constraints chosen uniformly at random among the rows of A.
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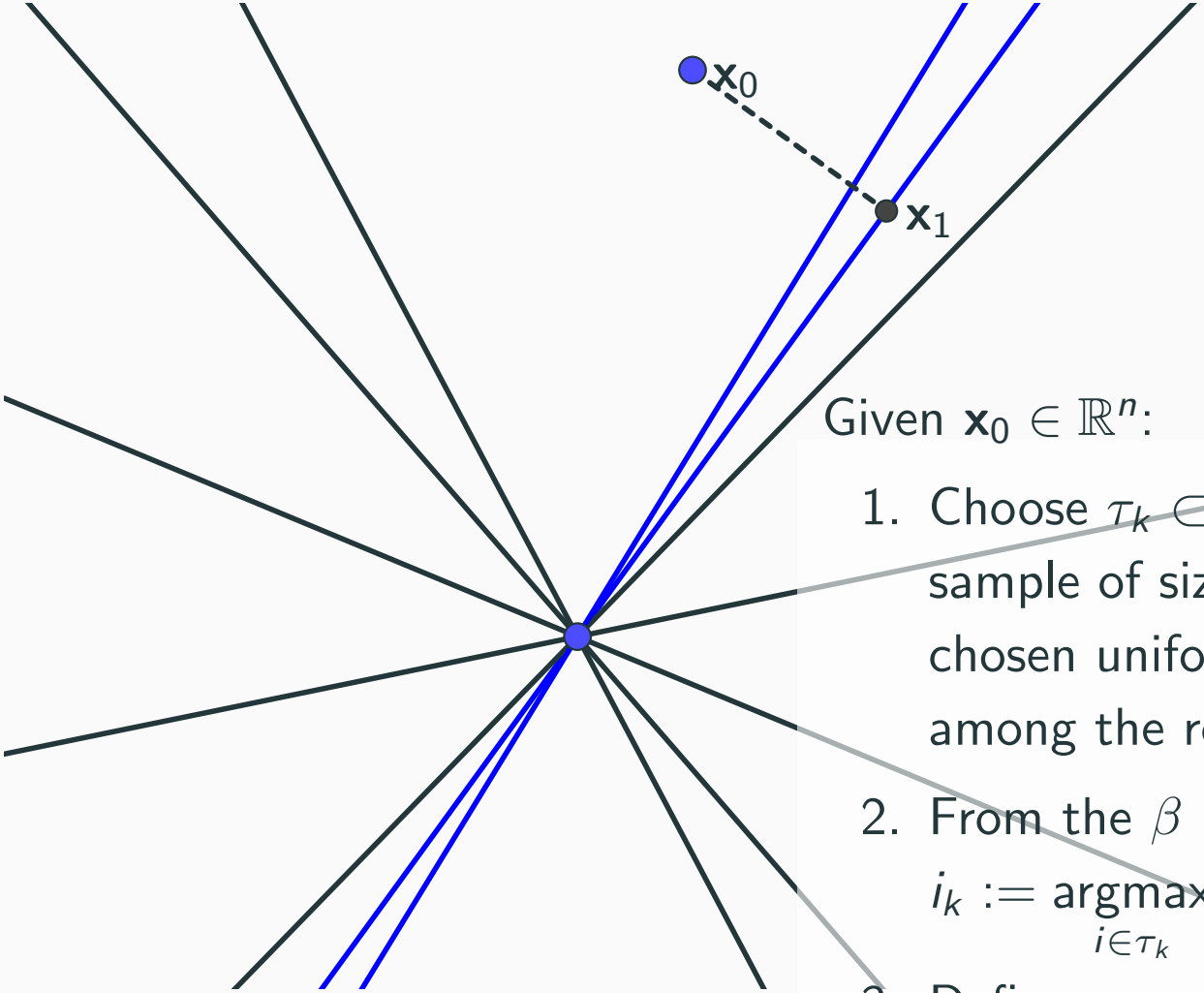

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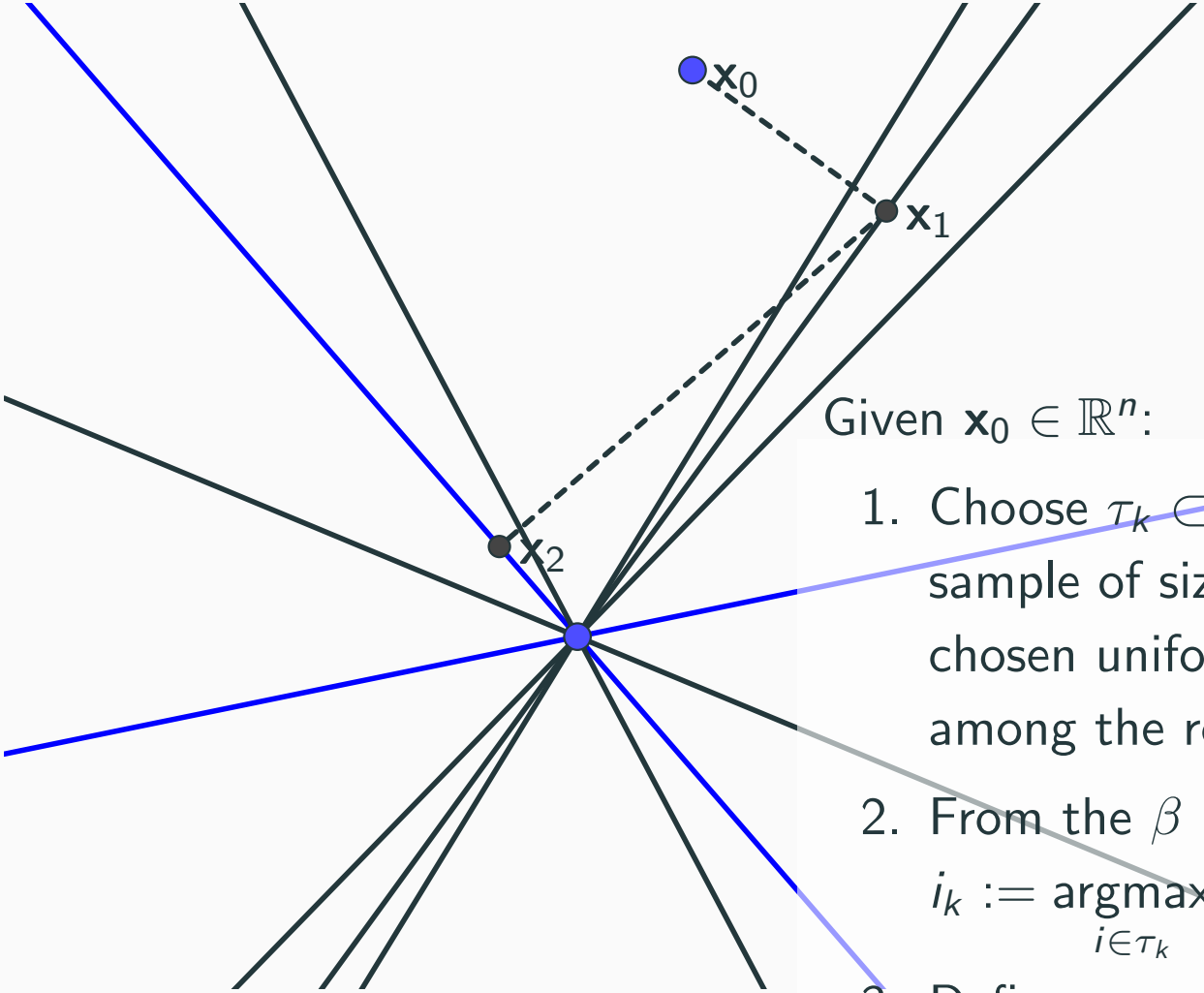

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Glimpse of HUGE Body of Literature

RK: [Strohmer-Vershynin '09], [Needell-Srebro-Ward '16]

Greedy: [Censor '81], [Nutini et al '16], [Bai-Wu '18], [Du-Gao '19]

Accel.: [Hanke-Niethammer '90], [Liu-Wright '16], [Morshed-Islam '19]

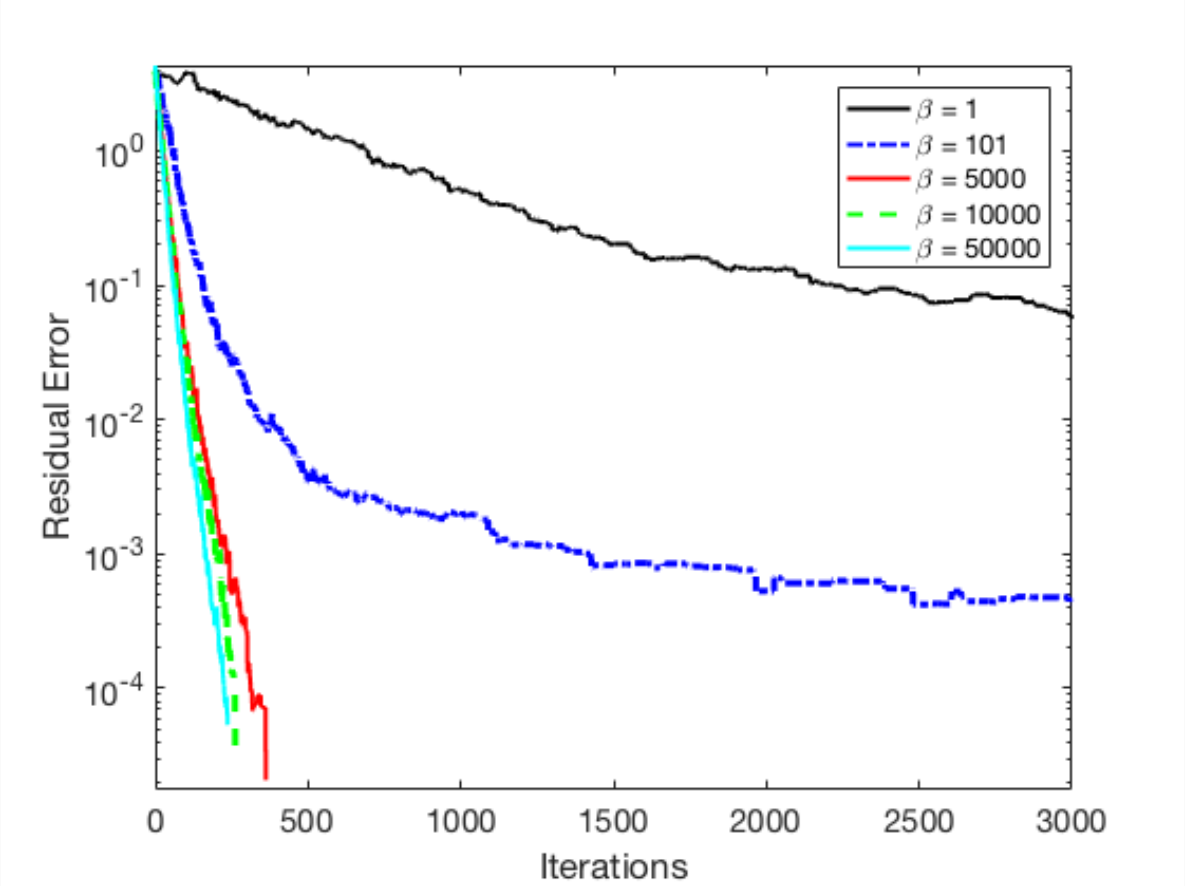
Block: [Popa et al '12], [Needell-Tropp '14], [Needell-Zhao-Zouzias '15],

Sketching: [Gower-Richtarik '15], [Needell-Rebrova '19]

Phase retrieval: [Tan-Vershynin '17], [Jeong-Güntürk '17]

LP: [Motzkin-Schoenberg '54], [Agmon '54], [Goffin '80], [Chubanov '12]

Experimental Convergence



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- 'faster' convergence for larger sample size

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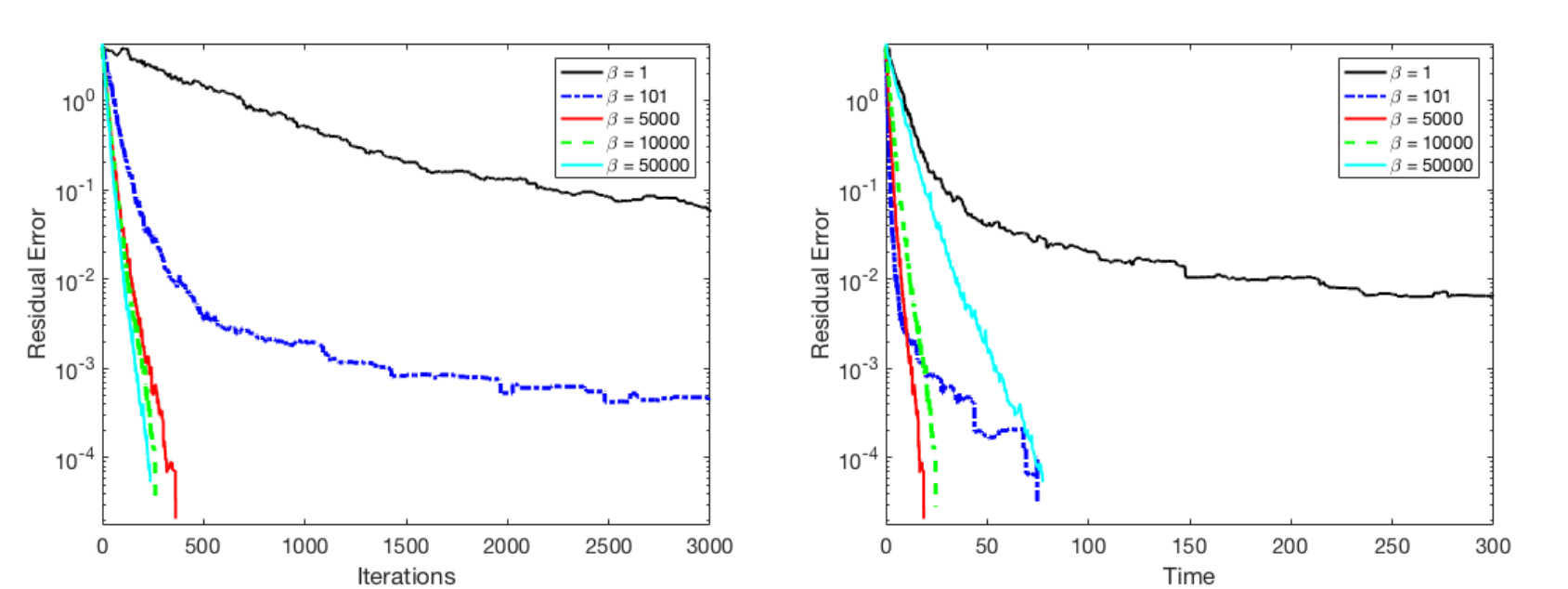

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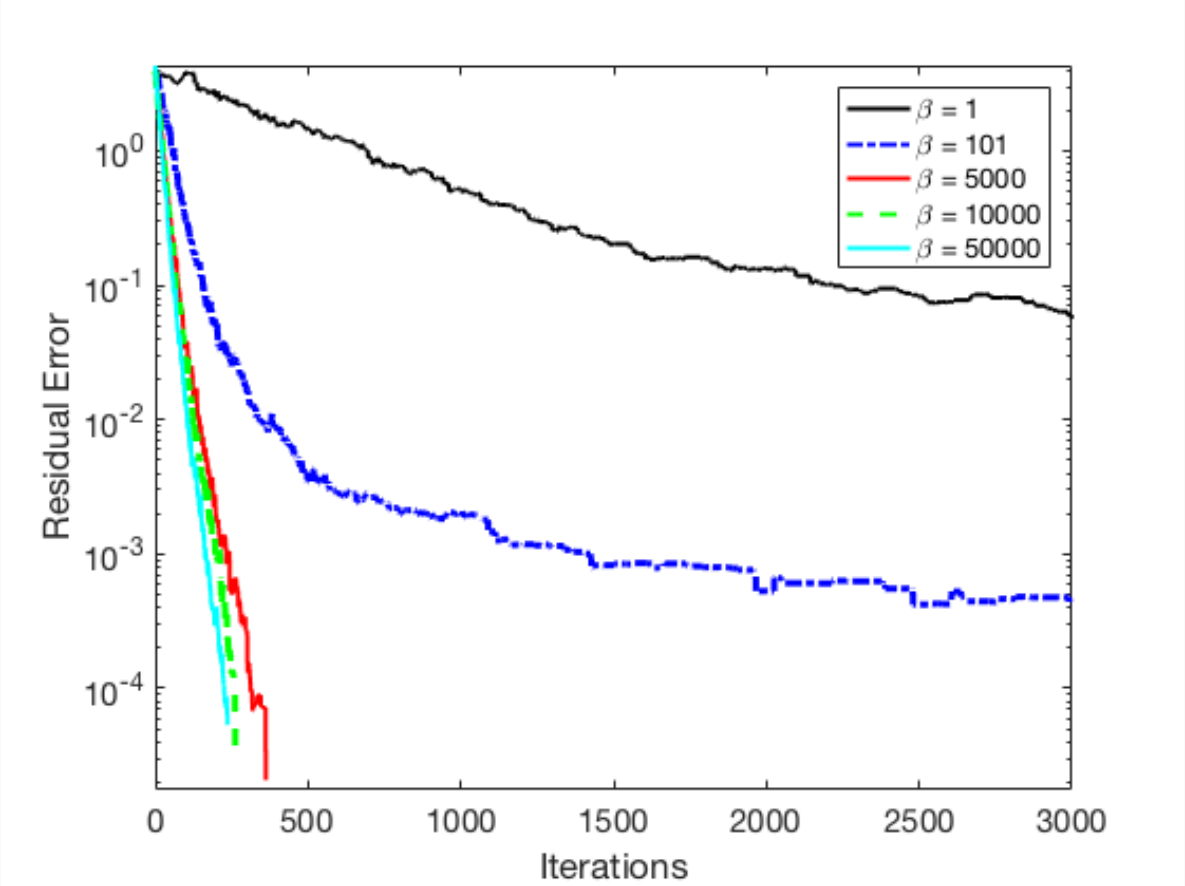

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Convergence Rates

Below are the convergence rates for the methods on a system, $A\mathbf{x} = \mathbf{b}$, which is consistent with unique solution $\mathbf{x}$, whose rows have been normalized to have unit norm.

- RK (Strohmer - Vershynin '09):

$$\mathbb{E}||\mathbf{x}_k - \mathbf{x}||_2^2 \leq \left(1 - \frac{\sigma_{\min}^2(A)}{m}\right)^k ||\mathbf{x}_0 - \mathbf{x}||_2^2$$

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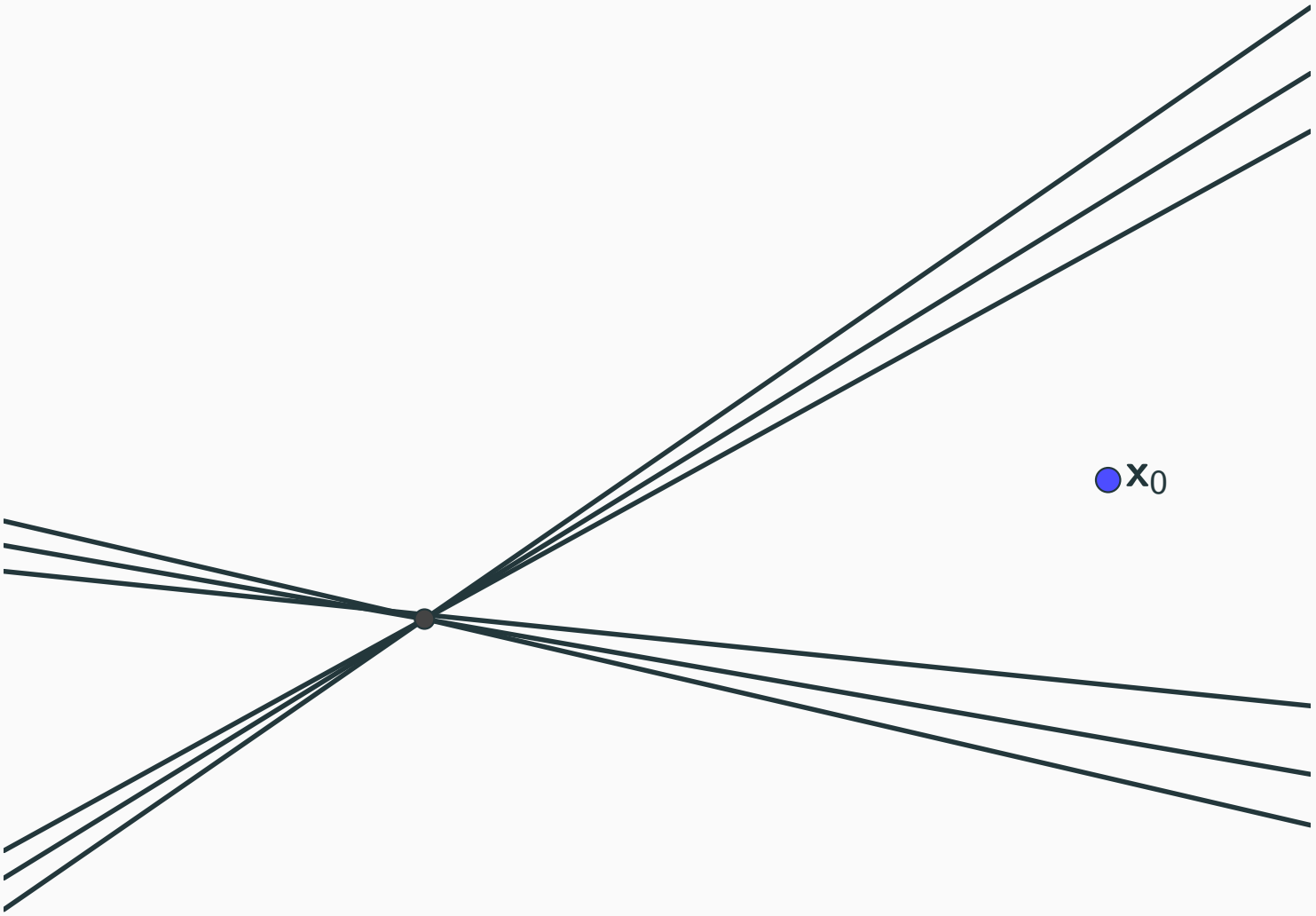
`Why are these all the same?`

A Pathological Example

Because.



Structure of the Residual

Several works have used sparsity of the residual to improve the convergence rate of greedy methods.

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However, not much sparsity can be expected in most cases. Instead, we'd like to use dynamic range of the residual to guarantee faster convergence.

$$\gamma_k := \frac{\|A\mathbf{x}_k - A\mathbf{x}\|^2}{\|A\mathbf{x}_k - A\mathbf{x}\|_\infty^2}$$

An Accelerated Convergence Rate

Theorem (H. - Needell '19)

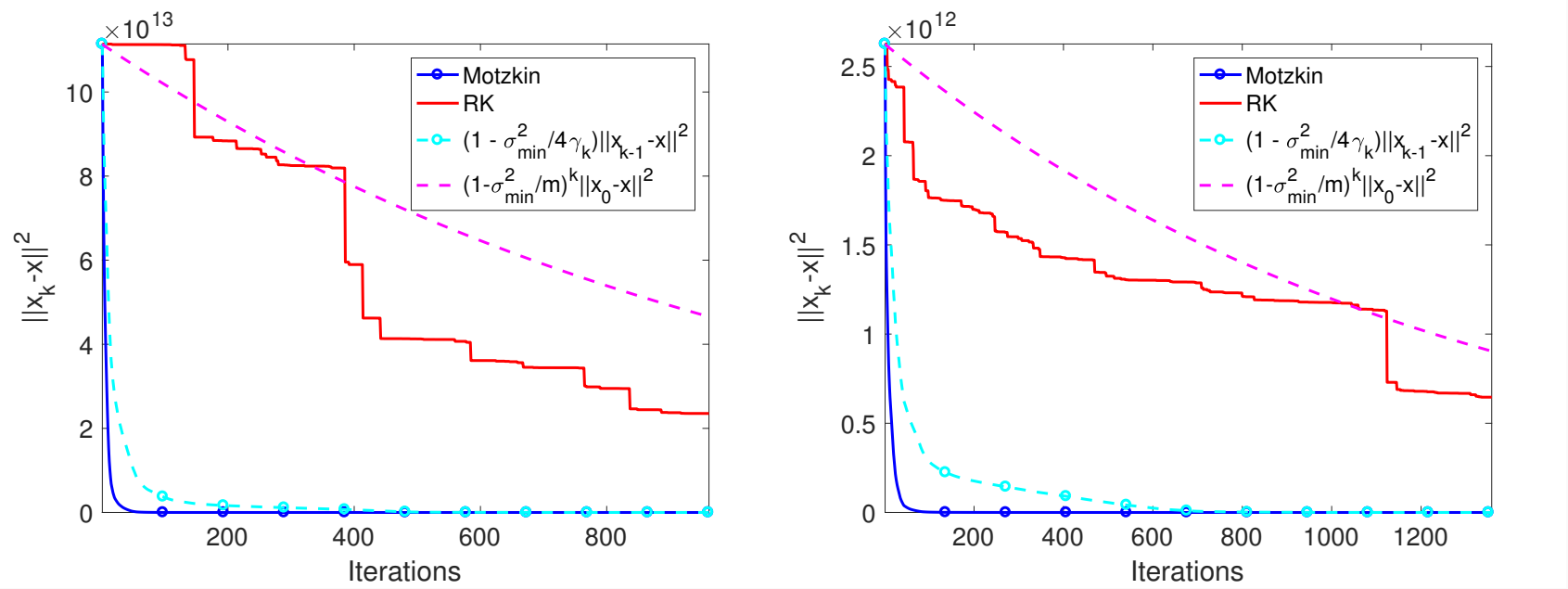
Let $\mathbf{x}$ *denote the solution of the consistent, normalized system* $A\mathbf{x} = \mathbf{b}$. *Motzkin's method exhibits the (possibly highly accelerated) convergence rate:*

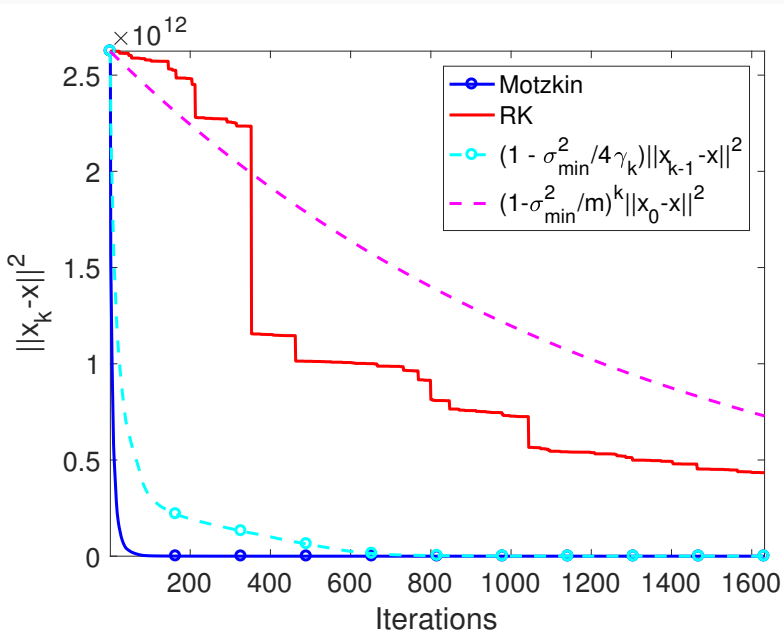
$$\|\mathbf{x}_T - \mathbf{x}\|^2 \leq \prod_{k=0}^{T-1} \left(1 - \frac{\sigma_{\min}^2(A)}{4\gamma_k}\right) \cdot \|\mathbf{x}_0 - \mathbf{x}\|^2$$

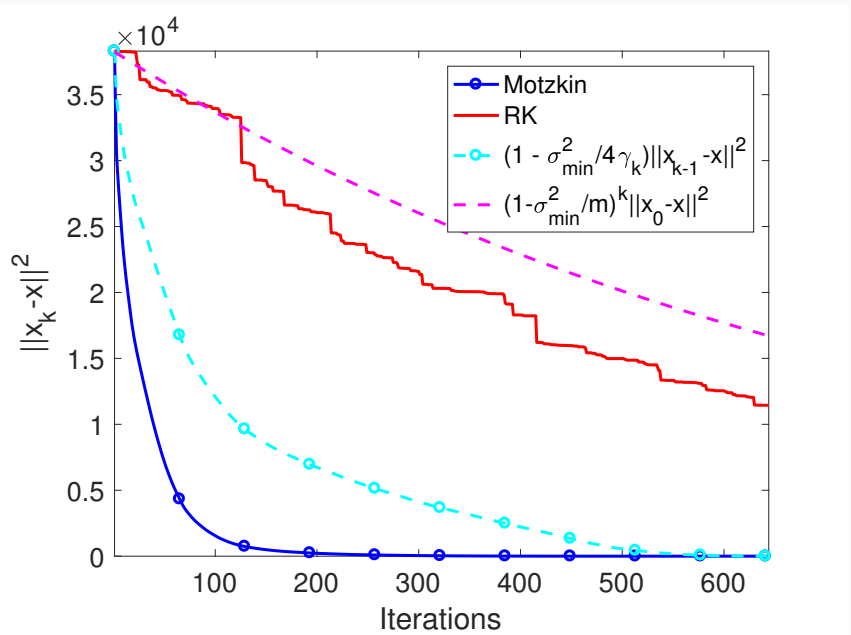
Here γ_k *bounds the dynamic range of the* k*th residual,* $\gamma_k := \frac{\|A\mathbf{x}_k - A\mathbf{x}\|^2}{\|A\mathbf{x}_k - A\mathbf{x}\|_\infty^2}$.

- ▷ improvement over previous result when $4\gamma_k < m$

Netlib **LP Systems**







Extending to SKM

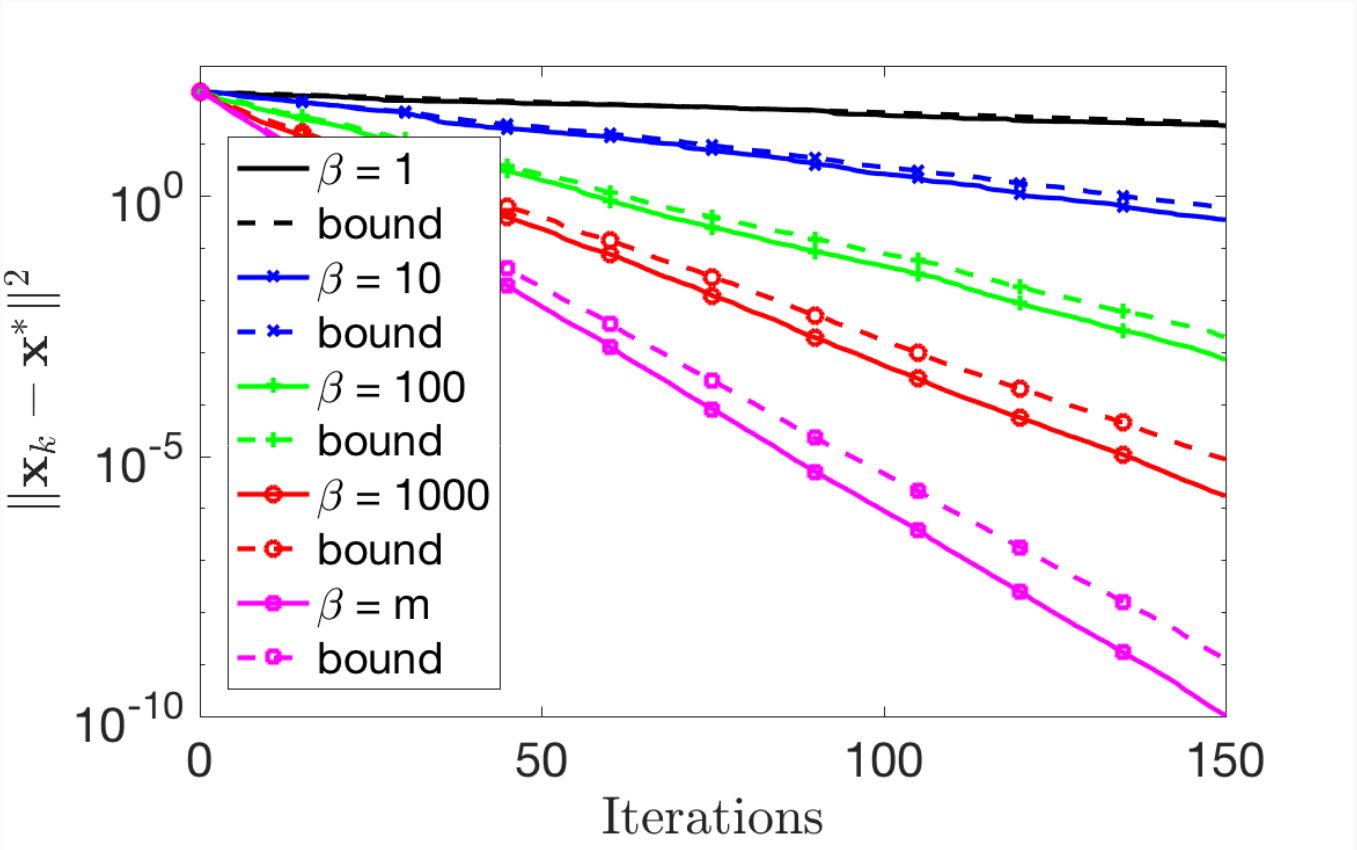
Corollary (H. - Ma 2019+)

Let A be normalized so $\|\mathbf{a}_i\|_2 = 1$ for all rows $i = 1, \ldots, m$. If the system $A\mathbf{x} = \mathbf{b}$ is consistent with the unique solution $\mathbf{x}^*$ then the SKM method converges at least linearly in expectation and the rate depends on the dynamic range of the random sample of rows of A, τ_j. Precisely, in the $j+1$st iteration of SKM, we have

$$\mathbb{E}_{\tau_j} \|\mathbf{x}_{j+1} - \mathbf{x}^*\|_2^2 \leq \left(1 - \frac{\beta \sigma_{\min}^2(A)}{\gamma_j m}\right) \|\mathbf{x}_j - \mathbf{x}^*\|_2^2$$

where $\gamma_j = \dfrac{\sum_{\tau_j \in \binom{[m]}{\beta}} \|A_{\tau_j}\mathbf{x}_j - \mathbf{b}_{\tau_j}\|_2^2}{\sum_{\tau_j \in \binom{[m]}{\beta}} \|A_{\tau_j}\mathbf{x}_j - \mathbf{b}_{\tau_j}\|_\infty^2}$.

Extending to SKM



- A is 50000×100 Gaussian matrix, consistent system
- bound uses dynamic range of sample of β rows

What can we say about γ_j?

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SKM	$1 - \frac{\beta\sigma^2_{\min}(A)}{m}$		$1 - \frac{\sigma^2_{\min}(A)}{m}$	
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Table 1: Contraction terms α such that $\mathbb{E}_{\tau_k}\|\mathbf{e}_k\|^2 \leq \alpha\|\mathbf{e}_{k-1}\|^2$.

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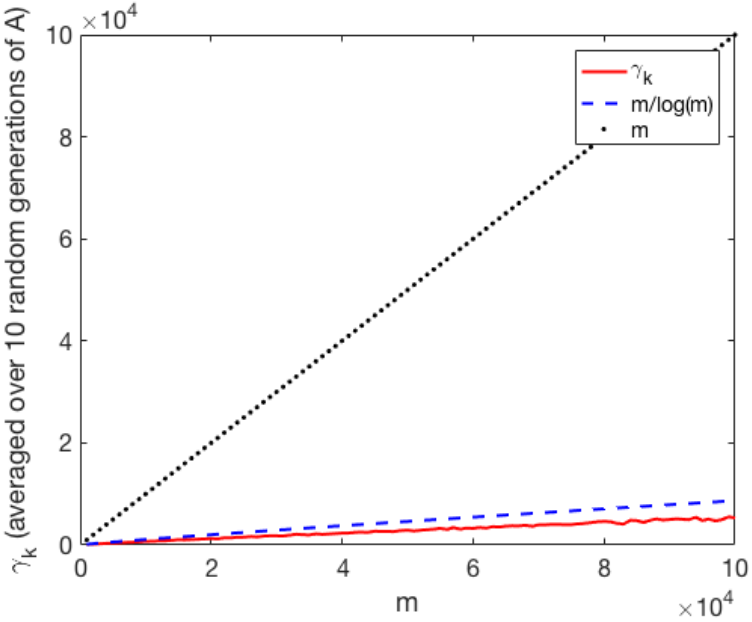
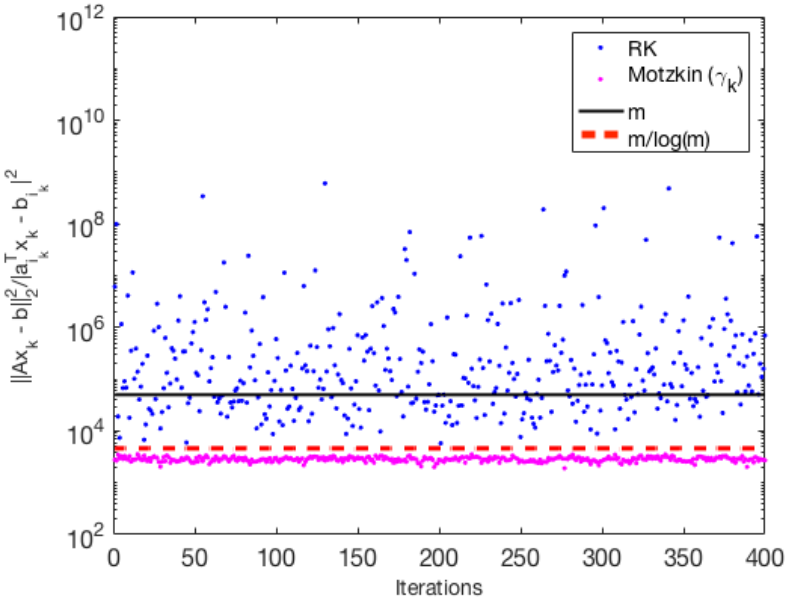
Nervous? $\gamma_k \geq \frac{\beta}{m}\sigma_{\min}^2(A)$ when A is row-normalized

γ_k: Gaussian systems





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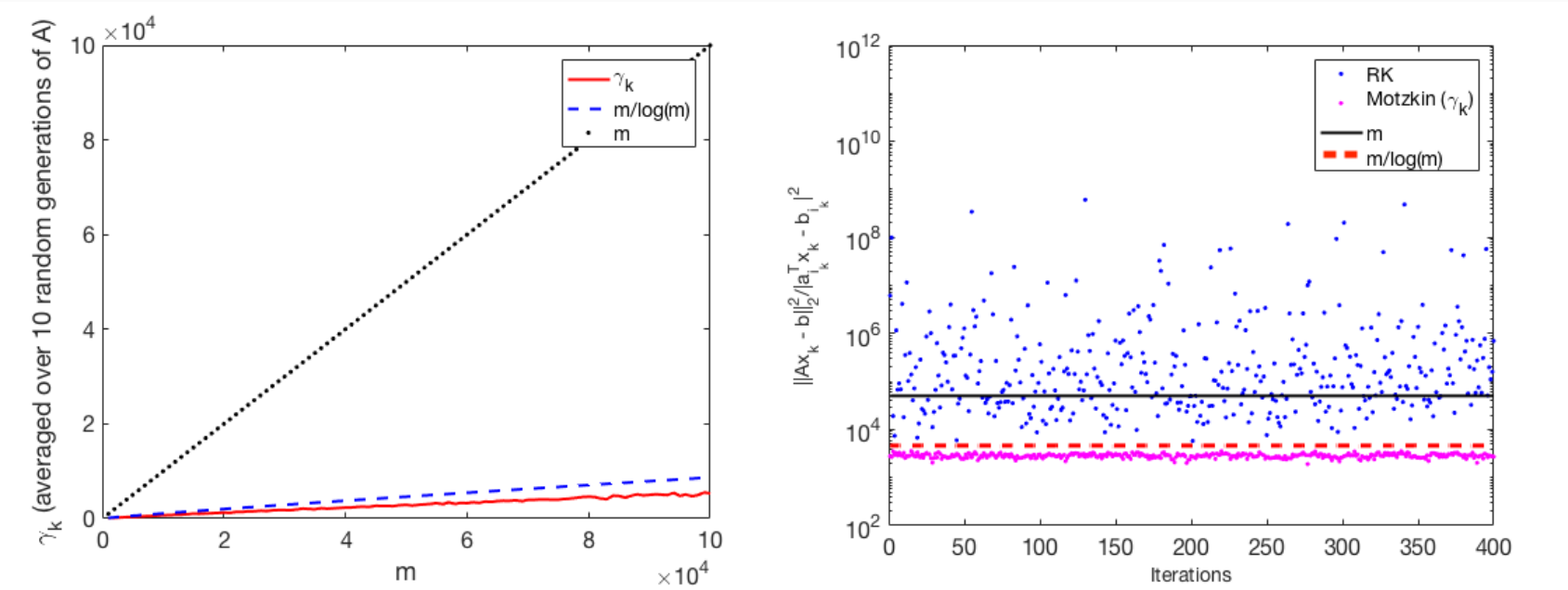
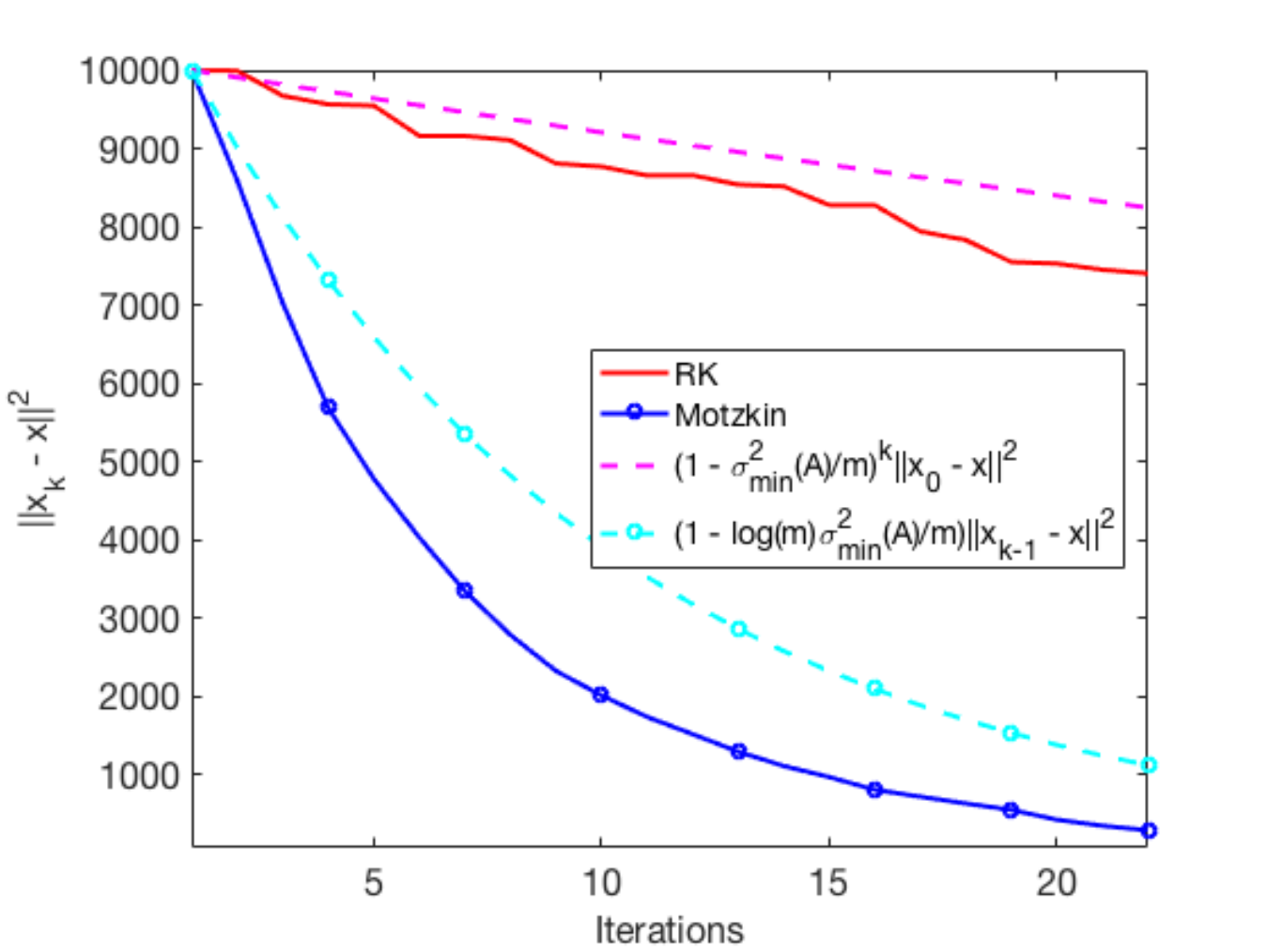


$$\gamma_k \lesssim \frac{n\beta}{\log \beta}$$

Gaussian Convergence



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γ_k: AC systems

$$\gamma_k \leq \frac{\beta \sum_{e_{ij} \in \mathcal{E}} (x_k^{(i)} - x_k^{(j)})^2}{|\mathcal{E}| (x_k^{(\beta_i)} - x_k^{(\beta_j)})^2}$$

β_i, β_j: nodes connected by the βth smallest residual

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  proportional to norm of selected row

Generalized SKM

Given $\mathbf{x}_0 \in \mathbb{R}^n$:

1. Choose $\tau_k \in \binom{[m]}{\beta_k}$ according to $p_{\mathbf{x}_{k-1}}$.
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Generalized Result

Theorem (H. - Ma 2019+)

Let $\mathbf{x}^*$ denote the unique solution to the system of equations $A\mathbf{x} = \mathbf{b}$. Then generalized SKM converges at least linearly in expectation and the bound on the rate depends on the dynamic range, γ_k of the random sample of β_k rows of A, τ_k. Precisely, in the kth iteration of generalized SKM, we have

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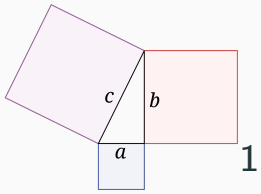
▷ If all rows of A have the same norm, then

$$\mathbb{E}_{\tau_k}\|\mathbf{x}_k - \mathbf{x}^*\|^2 \leq \left(1 - \frac{\beta_k \sigma_{\min}^2(A)}{\gamma_k \|A\|_F^2}\right)\|\mathbf{x}_{k-1} - \mathbf{x}^*\|^2.$$

Sketch of Proof

$$\|\mathbf{x}_k - \mathbf{x}^*\|^2 = \|\mathbf{x}_{k-1} - \mathbf{x}^*\|^2 - \frac{\|A_{\tau_k}\mathbf{x}_{k-1} - \mathbf{b}_{\tau_k}\|_\infty^2}{\|\mathbf{a}_{t(\tau_k,\mathbf{x}_{k-1})}\|^2}$$



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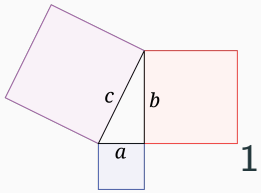

1

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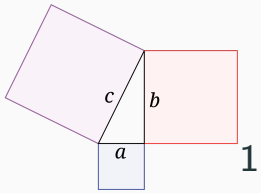

1

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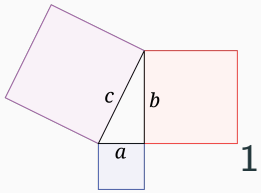

1

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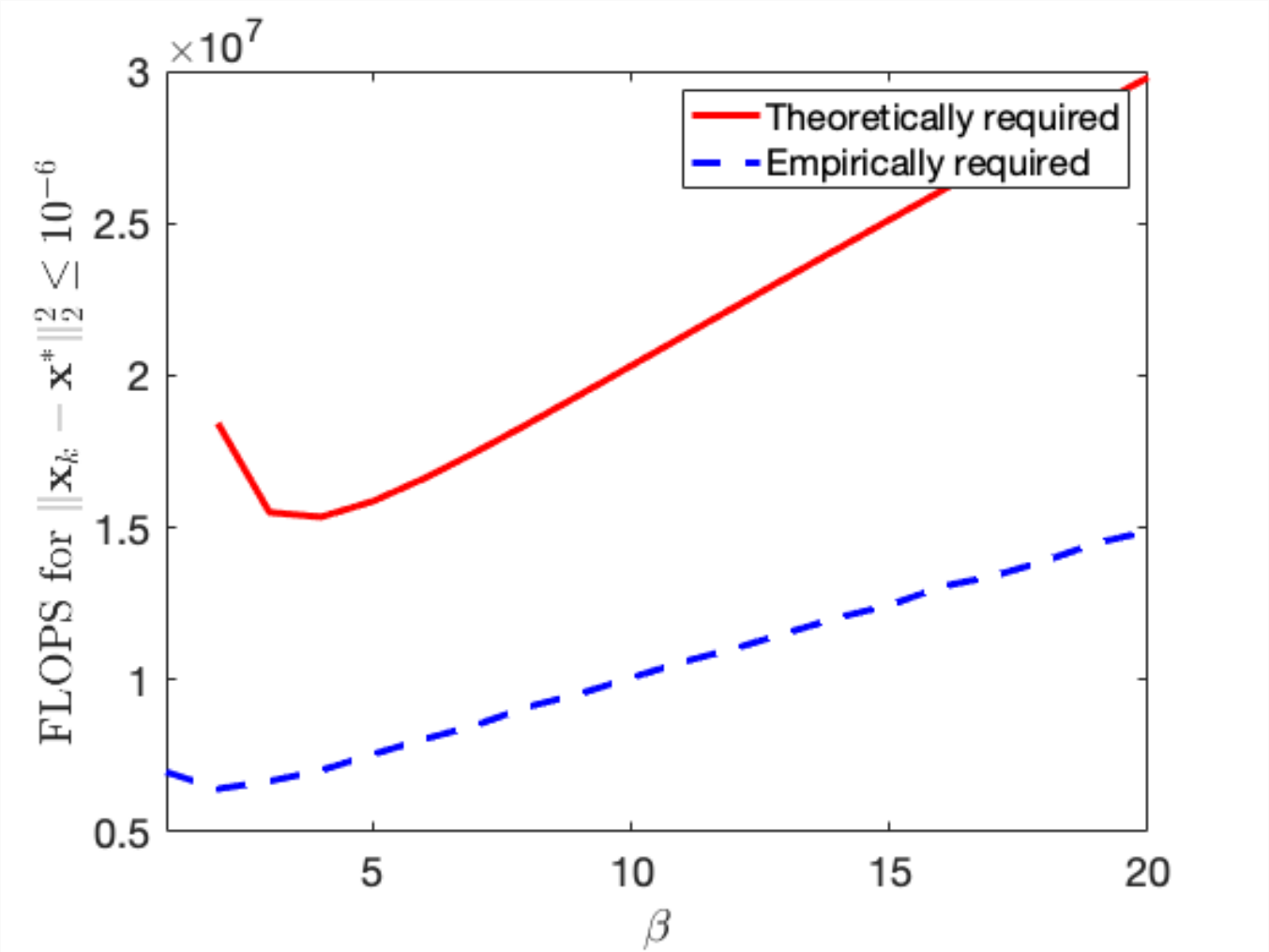

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- identify useful bounds on γ_k for other useful systems

Conclusions and Future Work

- presented a convergence rate for SKM which generalizes (and improves) results for RK and Motzkin

$$\mathbb{E}_{\tau_k}\|\mathbf{x}_k - \mathbf{x}^*\|^2 \leq \left(1 - \frac{\beta_k \binom{m}{\beta_k} \sigma_{\min}^2(A)}{\gamma_k m \sum_{\tau \in \binom{[m]}{\beta_k}} \|\mathbf{a}_{t(\tau, \mathbf{x}_{k-1})}\|^2}\right) \|\mathbf{x}_{k-1} - \mathbf{x}^*\|^2$$

- proved a result which "easily" yields results which illustrate SKM improvement for specific types of measurement matrices
- specialized our result to Gaussian matrices and AC systems

- identify useful bounds on γ_k for other useful systems
- identify optimal β of systems for which γ_k is known

Thanks for listening!

Questions?

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Block Kaczmarz